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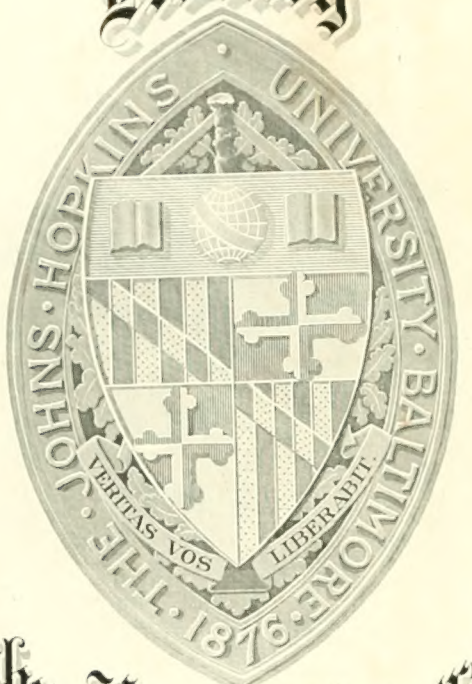

of the

Johns Hopkins University

A New Canonical Form of the Elliptic Integral

A Dissertation

submitted to the Board of University Studies of the Johns Hopkins University in Conformity with the Requirement for the Degree of Doctor of Philosophy.

by

Bessie Irving Miller

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The Hyperboloid of Revolution

Introduction

In the Vorlesungen "über die Theorie der elliptischen Modulfunctionen" by Klein and Fricke, all of the modular functions, and functions of $w = \frac{w_1}{w_2}$, when w_1 and w_2 are periods of the elliptic integral of the first kind, are closely examined. Moreover elliptic normal curves are discussed in connection with modular functions.

The paper here presented is intimately connected with various parts of Klein's deep investigations. Its purpose is three-fold; first, to deduce a general form of a Modular equation, that includes the particular forms obtained by Klein, secondly, to give geometric significance to some forms which appear in the

new canonical form of the elliptic integral used for the first part; thirdly, to map out the elliptic quartic curve, Δ_4, on Thomae's second space S_3.

Bianchi has worked with Δ_5 in S_2 and with Δ_5 in S_4. His methods however have not always been followed in the paper. In fact a fragment of the method of projecting the elliptic curve in hyperspace suggested by the character of the collineations by means of which it is transformed into itself, as a means of establishing a correspondence between the points in the curve and the points in the many domains is perhaps the most distinguishing feature in the discussion of the elliptic curves.

List of Abbreviations.

K. F. "Theorie der Thetafunctionen der Elliptischen Modulfunctionen" by Klein und Fricke.

B. "Über die homogenen ternären quadratischen Formen" by Luigi Bianchi in Mathematische Annalen, Band XLIX.

G. & Y. "Algebra of Invariants" by Grace and Young.

S. "Analytic Geometry of Three Dimensions" by Salmon. The fifth edition is referred to.

Chapter I.

(1) $\int R(y_1, y_2, y_3)\, dy$

§1. An Elliptic Integral of the First Kind.

If the $(n-3)$rd polar of a binary form $a_z^n \equiv (a_1 z_1 + a_2 z_2)^n$ is taken with respect to $t = \frac{z_1}{z_2}$ and the result multiplied by the form $(zt) \equiv (z_1 t_2 - z_2 t_1)$, then

$$u = \int \frac{z_2\, dz_1 - z_1\, dz_2}{\sqrt{a_t^{n-3}\, a_z^3\, (zt)}}$$

is an elliptic integral of the first kind. The expression under the radical, since it is a quartic in the variable $z = \frac{z_1}{z_2}$, has two invariants g_2 and g_3 (vide K.-F. I: pp. 13–15). Since g_2 is of the second degree and g_3 of the third degree in the coefficients of the quartic in z, g_2 is of degree $2(n-2)$ and g_3 of degree $3(n-2)$ in t. But $f \equiv a_t^n$ has one covariant, H, of degree $2(n-2)$ in t and one covariant, obtained by taking the first transvectant of f and H, $(f, H)^1$, which is of degree $3(n-2)$ in t. Therefore g_2 must become multiple of H; g_3, some multiple of $(f, H)^1$. Now the discriminant, Δ, of a quartic is expressible in terms of g_2^3 and g_3^2. Then if T is properly defined, the

following purpose may be written:

$$J : J-1 : 1 = g_2^3 : 27 g_3^2 : -.$$

The purpose of this chapter is to obtain g_2, g_3 and Δ expressed in terms of the transvectants of f, so that a syzygy between these forms may be found analogous to that existing in the case of the quartic.

§ 2. The invariant g_2.

$$\alpha_t^{n-3}\alpha_2^3 \cdot \beta_t^{n-3}\beta_2^3 \cdots \zeta_t^{n-3}\zeta_2^3 \cdot T^{n-3}$$

$(\alpha_t^{n-3}\alpha_2^3(zt), \beta_t^{n-3}\beta_2^3(zt))^4$ equals $2g_2$ according to Aronhold's definition of g_2. But

$$(\alpha_t^{n-3}\alpha_2^3(zt), \beta_t^{n-3}\beta_2^3(zt))^4 = \alpha_t^{n-3}\beta_t^{n-3}(\alpha_2^3(zt), \beta_2^3(zt))^4$$

The fourth transvectant is calculated by operating with $\alpha_2^3(zt)$ on $\beta_2^3(zt)$, where $\alpha_1 z_1 + \alpha_2 z_2$ as an operator means $\alpha_1 \frac{\partial}{\partial z_2} - \alpha_2 \frac{\partial}{\partial z_1}$, and dividing the result by $4!$. The fourth transvectant, obtained by these processes, equals $-\frac{18}{4!}(\alpha\beta)^2 \alpha_t^{n-2}\beta_t^{n-2}$.

Hence $g_2 = -\frac{3}{8}(\alpha\beta)^2 \alpha_t^{n-2}\beta_t^{n-2} = -\frac{3}{8}It$.

§ 3. The form g_3.

Let $g \equiv \alpha_t^{n-3}\,\alpha_2^3(zt) \equiv \beta_t^{n-3}\,\beta_2^3(zt) \equiv \gamma_t^{n-3}\,\gamma_2^3(zt)$ and let H_g represent the Hessian of g. Then $(H_g, g)^4$ equals $4\,g_3$ according to Aronhold's definition of g_3.

If the method of Grace and Young is used (see G. & Y. § 43, 49), it is found that

$$H_g = \tfrac{1}{2}(\alpha\beta)^2\,\alpha_t^{n-3}\,\beta_t^{n-3}\,\alpha_2\beta_2(zt)^2;$$

$$(H_g, g)^4 = \tfrac{6}{4!}(\alpha\beta)^2(\gamma\alpha)^2\,\alpha_t^{n-3}\,\beta_t^{n-2}\,\gamma_t^{n-1}.$$ Hence

$$g_3 = \tfrac{1}{16}(\alpha\beta)^2(\gamma\alpha)\,\alpha_t^{n-3}\,\beta_t^{n-2}\,\gamma_t^{n-1} = \tfrac{1}{16}(f, H)'.$$

Vide G. & Y. p. 47 ex iii for method of obtaining the explicit expression of such a covariant $= (f, H)'$.

§ 4. The Expression for $T = (g, g_3)^2$.

Let

$$f \equiv \alpha_t^n \equiv \beta_t^n \equiv \gamma_t^n \equiv \delta_t^n \equiv \varepsilon_t^n \equiv \eta_t^n.$$

$$H \equiv (\alpha\beta)^2\,\alpha_t^{n-2}\,\beta_t^{n-2} \equiv (f, f)^2;$$

$$\bar{t} \equiv (\alpha\beta)^2(\gamma\alpha)\,\alpha_t^{n-3}\,\beta_t^{n-2}\,\gamma_t^{n-1} \equiv (f, H)';$$

$$i \equiv (\alpha\beta)^4\,\alpha_t^{n-4}\,\beta_t^{n-4} \equiv (f, f)^4;$$

$$J = (x\beta)^2 (\beta\gamma)^2 (\gamma x)^2 \lambda_t^{n-4} \mu_t^{n-4} \nu_t^{n-4} = (f,f)^2,$$
$$J_2 = (x\beta)^6 \lambda_t^{n-6} \mu_t^{n-6} = (f,f)^6.$$

Any J_2 and J_3, J_2 and J_3 may be expressed in terms of H and $\bar{T}$. But since $\bar{T}$ is an $(n-4)$-ic, the expression is unwieldy (Clebsch, Alg. 378).

$$-2\bar{T}^2 = (f,f)^2 H^2 + (H,H)^2 f^2 - 2(f,f)^2 f'' H.$$

I: $-2\bar{T}^2 - H^3 = (H,H)^2 f^2 - 2(f,H)^2 f^4$

Let $\Delta' = 2\bar{T}^2 + H^3$, $J = \frac{H^3}{\Delta'}$. Then $J-1 = -\frac{2\bar{T}^2}{\Delta'}$.

The following proportion may now be written:

$$J : J-1 : 1 = H^3 : -2\bar{T}^2 : \Delta'.$$

This is essentially a relation between the Hessian and Jacobian of an n-ic similar to the one existing between between the invariants J_2 and J_3 of a quantic. Δ' which corresponds to Δ of the quantic is by I expressible in terms of transvectants which are reducible. Therefore, before any discussion of the $(n-3)^{ic}$ is given, the reduction of $(H,f)^2$ and $(f,H)^4$ is performed

§5. Auxiliary Expressions.

For the reduction of $(-\beta, H)^2$ and $(+, H)^2$ it is convenient to have for reference several auxiliary expressions. In this work a subscript is used to indicate a polar: - e.g. H_1 means the first polar of H.

$$H_1 = \frac{n-2}{2n-4}(\alpha\beta)^2\{\alpha_t^{n-3}\beta_t^{n-2}\alpha_s + \alpha_t^{n-2}\beta_t^{n-3}\beta_s\}.$$

$$H_2 = \frac{1}{2n-5}(\alpha\beta)^2\{(n-3)\alpha_t^{n-4}\beta_t^{n-2}\alpha_s^2 + (n-2)\alpha_t^{n-3}\beta_t^{n-3}\alpha_s\beta_s\}.$$

$$f_1 = \gamma_t^{n-1}\gamma_s.$$

$$f_2 = \gamma_t^{n-2}\gamma_s^2$$

$$i_1 = (\alpha\beta)^4\alpha_t^{n-5}\beta_t^{n-4}\alpha_s.$$

$$i_2 = \frac{(\alpha\beta)^4}{2n-9}\{(n-5)\alpha_\tau^{n-6}\beta_\tau^{n-4}\alpha_s^2 + (n-4)\alpha_\tau^{n-5}\beta_\tau^{n-5}\alpha_s\beta_s\}.$$

The three following identities are given by Grace and Young (Alg. of Inv. § 22).

(a). $(\alpha\beta)\gamma_t = (\gamma\beta)\alpha_t - (\gamma\alpha)\beta_t$ or $(\beta\gamma)\alpha_t + (\alpha\gamma)\beta_t = -(\alpha\beta)\gamma_t$.

(b). $2(\alpha\beta)(\alpha\gamma)\beta_\tau\gamma_t = (\alpha\beta)^2\gamma_t^2 + (\alpha\gamma)^2\beta_t^2 - (\beta\gamma)^2\alpha_t^2$.

(c). $(\beta\gamma)^4\alpha_t^4 + (\gamma\alpha)^4\beta_\tau^4 + (\alpha\beta)^4\gamma_t^4 = 2\{(\alpha\beta)^2(\alpha\gamma)^2\beta_\tau^2\gamma_\tau^2 + (\beta\gamma)^2(\beta\alpha)^2\gamma_t^2\alpha_t^2 + (\gamma\alpha)^2(\gamma\beta)^2\alpha_\tau^2\beta_\tau^2\}$.

Instead of (b) the following is sometimes useful.

(b'). $2(\gamma\alpha)(\gamma\beta)\alpha_t\beta_t - (\gamma\alpha)^2\beta_t^2 - (\gamma\beta)^2\alpha_t^2 - (\alpha\beta)^2\gamma_t^2$.

Several expressions will be seen to vanish.

$H = (\beta\gamma)^2(\alpha\beta)(\alpha\gamma)\alpha_t^{n-2}\beta_t^{n-3}\gamma_t^{n-3}$.

Multiply by the second form of (a) $(\alpha\beta)(\alpha\gamma)\beta_t^{n-2}\gamma_t^{n-2}$.

Then $(\beta\gamma)^2(\alpha\beta)(\alpha\gamma)\alpha_t^{n-2}\beta_t^{n-3}\gamma_t^{n-3}$ or

$$H = \alpha_t^{n-3}\beta_t^{n-3}\gamma_t^{n-3}\{(\beta\alpha)^2(\beta\gamma)(\gamma\alpha)\gamma_t - (\gamma\alpha)^2(\gamma\beta)(\alpha\beta)\beta_t\}$$
$$= 2\alpha_t^{n-3}\beta_t^{n-3}\gamma_t^{n-3}\{(\beta\alpha)^2(\beta\gamma)(\gamma\alpha)\gamma_t\}$$
$$= 2\alpha_t^{n-3}\beta_t^{n-3}\gamma_t^{n-3}\{(\beta\gamma)^2(\beta\alpha)(\alpha\gamma)\alpha_t\}$$

since β and γ are equivalent symbols, and α and γ are equivalent symbols.

Therefore $H - 2H = 0$ or $H = 0$.

Let $B = (\alpha\beta)^2(\alpha\gamma)^2\beta_t^2\gamma_t^2\alpha_t^{n-4}\beta_t^{n-4}\gamma_t^{n-4}$.

Consider determinants only. By identity (c)

$$(\alpha\beta)^2(\alpha\gamma)^2 = \tfrac{1}{2}(\beta\gamma)^4 - (\beta\gamma)^2\{2(\alpha\beta)(\alpha\gamma) + (\beta\gamma)^2\}$$
$$= \tfrac{1}{2}(\beta\gamma)^4 - 2(\beta\gamma)^2(\alpha\beta)(\alpha\gamma).$$

The last expression is obtained by using (b').

$$\therefore B = \tfrac{1}{2}(\beta\gamma)^4\alpha_t^n\beta_t^{n-4}\gamma_t^{n-4} - 2(\beta\gamma)^2(\alpha\beta)(\alpha\gamma)\alpha_t^{n-2}\beta_t^{n-3}\gamma_t^{n-3}$$
$$= \tfrac{1}{2}i f + H = \tfrac{1}{2}i f.$$

Let $C=(\alpha\beta)^4(\alpha\gamma)(\beta\gamma)\alpha_t^{n-5}\beta_t^{n-5}\gamma_t^{n-2}$

Instead of $(\alpha\beta)^2\gamma_t^2$ use the expression for it obtained from (b') and consider the determinants only. Then it may be shown that

$(\alpha\beta)^4(\alpha\gamma)(\beta\gamma) = -(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2$.

$\therefore C = -(\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)\alpha_t^{n-4}\beta_t^{n-4}\gamma_t^{n-4} = -j$.

Incidentally in finding the value of C it is deduced that

$\therefore G = (\alpha\beta)^2(\beta\gamma)^2(\alpha\gamma)(\beta\gamma)\alpha_t^{n-3}\beta_t^{n-3}\gamma_t^{n-4} = -\frac{1}{2}k$

Let $E = (\alpha\beta)^4(\alpha\gamma)^2\alpha_t^{n-6}\beta_t^{n-4}\gamma_t^{n-2}$. Instead of $(\alpha\beta)^2(\alpha\gamma)^2\beta_t^2\gamma_t^2$ use the expression for it obtained from (c) and consider the determinants only. It may then be shown that

$(\alpha\beta)^4(\alpha\gamma)^2 = \frac{1}{2}(\alpha\beta)^6 - (\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2$.

$\therefore E = \frac{1}{2}(\alpha\beta)^6\alpha_t^{n-6}\beta_t^{n-6} - (\alpha\beta)^2(\beta\gamma)^2(\gamma\alpha)^2\alpha_t^{n-4}\beta_t^{n-4}\gamma_t^{n-4}$.

$= \frac{1}{2}kf - j$.

Let $F = (\alpha\beta)^4(\alpha\delta)(\gamma\delta)\alpha_t^{n-5}\beta_t^{n-4}\gamma_t^{n-1}\delta_t^{n-2}$.

Re-express $(\alpha\delta)\gamma_y$ by means of (a) and consider the determinants only.

Then $(\alpha\beta)^4(\alpha\gamma)(\beta\delta) - (\alpha\beta)^3(\alpha\gamma)^2(\beta\delta)^2 - (\alpha\beta)^2(\alpha\gamma)^4(\beta\delta)^2(\alpha\beta)$.

$\therefore F = \frac{1}{2}(\alpha\beta)^4(\beta\delta)^2 \alpha_t^{n-4}\beta_t^{n-4}\gamma_t^{n-2}\delta_t^{n-2} = \frac{1}{2}\cdot H$.

It would be convenient to have an expression for $(i,f)^2$. From it it is found that

$$(i,f)^2 = \frac{1}{2n-4}\{(n-5)\Sigma^2 + (n-4)C\}$$
$$= \frac{n-5}{2(2n-4)}kf - j.$$

96. Reduction of $(f,H)^2$.

$(f,H)^2 = (\gamma\delta)^2 \gamma_t^{n-2}\delta_t^{n-2} h_t^{2n-4}$ where

$h_t^{2n-4} = (\alpha\beta)^2 \alpha_t^{n-2}\beta_t^{n-2}$. From the value of H_2 given in § 5 it may be shewn that

$$(f,H)^2 = \frac{1}{2n-5}(\alpha\beta)^2\{(n-3)(\alpha\gamma)^2\alpha_t^{n-4}\gamma_t^{n-2}\delta_t^{n-2}$$
$$+ (n-2)(\alpha\gamma)(\alpha\delta)(\beta\gamma)\alpha_t^{n-3}\beta_t^{n-3}\gamma_t^{n-2}\delta_t^{n-2}\}.$$

If (b') is applied to the last term on the right hand side and use made of the fact that α and β are equivalent symbols, it is found that

$$(f,H)^2 = B - \frac{n-2}{2(2n-5)}\cdot f$$
$$= \frac{1}{2}\cdot f - \frac{n-2}{2(2n-5)}\cdot f$$
$$= \frac{n-3}{2(2n-5)}\cdot f.$$

§7. Reduction of $(H, H)^2$.

The reduction of $(H, H)^2$ is accomplished by means of Gordan's series (vide Sy. §54). Since $(H, H)^2 = ((f, f)^2, H)^2$ we write the series $\begin{pmatrix} f & f & H \\ x & x & 2n-4 \\ 0 & 2 & 2 \end{pmatrix}$. Then

$$\sum \frac{\binom{n}{i}\binom{2}{i}}{\binom{2n-3}{i}} (f, f)^{2+i}, H)^{2-i}$$

$$\therefore \sum_i \frac{\binom{2n-6}{i}\binom{2}{i}}{\binom{3n-7}{i}} ((f, H)^{2+i}, f)^{2-i},$$

that is

$$\text{II}: (H, H)^2 + \frac{n-2}{2(2n-5)} i H = ((f, H)^2, f)^2 + \frac{4(n-3)}{3n-8} ((f, H)^3, f)^1 + \frac{2(2n-7)}{3(3n-10)} ((f, H)^4, f)^0.$$

It is now necessary to reduce $(f, H)^2$, $(f, H)^3$, and $(f, H)^4$ in order to obtain the right hand side of II in terms of the transvectants given in §4.

$$(f, H)^2 = \frac{n-3}{2(2n-5)} i f. \quad \text{(vide §4)}$$

$(f, H)^3$ is calculated by means of the series $\begin{pmatrix} f & f & f \\ n & n & n \\ 0 & 2 & 3 \end{pmatrix}$.

The result is $(f, H)^3 = \frac{n-4}{2(2n-5)} (-f, i)^1$.

$(f, H)^4 = (f, (f, f)^2)^4$ is calculated by means of the series $\begin{pmatrix} f & f & f \\ n & n & n \\ 3 & 4 & 2 \end{pmatrix}$. The result is

$$(H, f)^4 = -\frac{n-1}{2n-5} (i, f)^2 + \frac{(3n-10)(n-5)}{4(2n-7)(2n-9)} k f.$$

If the value of $(i, f)^2$ obtained in § 5 is introduced, then $(H, f)^4 = \frac{n-1}{2n-5} \{ \} + \frac{(n-4)(n-5)}{4(2n-5)(2n-7)} k f$.

The terms in the right hand side of II may now be calculated

$((f, H)^2, f)^2 = \frac{n-3}{2(2n-5)} (if, f)^2$. To obtain the right hand member explicitly use the method given by Grace and Young. § 47. Then

$$(if, f)^2 = \frac{3n-3!}{3n-8!} \{ n(n-1)(H + 2n(2n-8)(\alpha\beta)^4(\alpha\delta)(\gamma\delta) \alpha_t^{n-5} \beta_t^{n-4} \gamma_t^{n-1} \delta_t^{n-2} + (2n-8)(2n-9)(i, f)^2 f \}.$$

In making use of some of the expressions given in § 5 the result finally obtained is

$$((f, H)^2, f)^2 = \frac{1}{2(3n-8)(2n-5)} \{ n(n-4)(H - \frac{2(n-4)(2n-9)}{3} kf + \frac{(n-4)(n-5)}{3} i^2 f^2 \}.$$

$(\{f,H\}_3 f)' = \frac{n-4}{2(2n-5)}(\{f,\cdot\}',f)'$ is next to be calculated (under § 4. p. 78. XIX).

$$(\{f,\cdot\}',f)' = -\frac{2n-4}{3n-10}\left\{\frac{n-3}{2(2n-4)}kf-1\right\}f+\frac{1}{2}\cdot H.$$

By substituting the values of the terms in the right hand member of II it is easy to deduce that

$$(H,H)^2 = -\frac{1}{2(2n-5)}\cdot H + \frac{1}{3}Jf.$$

It is interesting to note that the term kf^2 has zero coefficient, i.e. the term involving k, a form which has no correspondent in the case of a quartic, does not appear in the final result.

§ 8. The Syzygy.

In § 4 the relation between $\bar{T}$ and H is given by

$$\text{I}: \quad -2\bar{T}^2 - H^3 = (H,H)^2 f^2 - 2(f,H)^2 fH.$$

I introduce the values of the terms in the right hand member of I as they were determined in §

and §7. Then

III: $2\bar{T}^2 + H^3 = f^2\{\frac{1}{2}iH - \frac{1}{3}jf\}$.

The projective genus in § 4 may now be written

Π: $J : J-1 : 1 = H^3 : -2\bar{T}^2 : f^2\{\frac{1}{2}iH - \frac{1}{3}jf\}$ or

Γ: $J : J-1 : 1 = 6H^3 : -12\bar{T}^2 : f^2\{3iH - 2jf\}$.

But according to § 2 and § 4, $g_2 = -\frac{3}{5}H$ and $g_3 = \frac{1}{16}\bar{T}$. Therefore Γ becomes

Γ': $J : J-1 : 1 = g_2^3 : 27g_3^2 : g_2^3 - 27g_3^2$,

the usual form of the syzygy existing between the g_2, g_3, and Δ of a quantic. Hence Γ is the syzygy desired between the Hessian and Jacobian of an n-ic. It also shows that as g_2 and g_3 correspond to H and $\bar{T}$ so does Δ correspond to Δ'.

§9. Discussion of the Roots of the Syzygy V.

For a given J, V is an algebraic equation of degree $6(n-2)$, with roots that are algebraic functions of J. It may be mapped on a $6(n-2)$-

...hantic Riemann surface. The values of σ giving the branch points of the surface will be determined after the k-tuple roots of I have been found.

A k-tuple root of I is a $(k-1)$-tuple root of $(H^3, \bar{T}^2)'$. In this connection the evaluation of $(H^3, \bar{T}^2)'$ is of interest

$$H = (\alpha\beta)^2 \alpha_t^{n-2} \beta_t^{n-2} = h_t^{2(n-2)}.$$

$$\bar{T} = (\delta\varepsilon)^2 (\eta\delta)\, \delta_t^{n-3} \varepsilon_t^{n-2} \eta_t^{n-1} = \bar{T}_t^{3(n-2)}.$$

Following the method given by Grace and Young (Ch. I, § 4) we obtain the transvectant

$$(H^3, \bar{T}^2)' = \frac{H^2 \cdot \bar{T} (\alpha\beta)^2 (\delta\varepsilon)^2 (\eta\zeta)}{6(n-2)} \left\{ \alpha_t^{n-3} \beta_t^{n-1} \right.$$

$$\left[(n-3)(\alpha\zeta)\, \beta_t \delta_t^{n-4} \varepsilon_t^{n-2} \eta_t^{n-1} + \right.$$

$$(n-2)(\alpha\varepsilon)\, \beta_t \delta_t^{n-3} \varepsilon_t^{n-3} \eta_t^{n-1} +$$

$$(n-1)(\alpha\eta)\, \beta_t \delta_t^{n-3} \varepsilon_t^{n-2} \eta_t^{n-2} +$$

$$(n-3)(\beta\delta)\, \alpha_t \delta_t^{n-4} \varepsilon_t^{n-2} \eta_t^{n-1} +$$

$$(n-2)(\beta\varepsilon)\, \alpha_t \delta_t^{n-3} \varepsilon_t^{n-3} \eta_t^{n-1} +$$

$$\left. (n-1)(\beta\eta)\, \alpha_t \delta_t^{n-3} \varepsilon_t^{n-2} \eta_t^{n-2} \right] \Big\}.$$

If $(H, \bar{T})'$ is calculated and the values substituted

in the above, then

$$(H^3, \bar{T}^2)' - H^3 \bar{T} (H, \bar{T})'$$
$$= H^3 \bar{T} f \left\{ \tfrac{n-3}{3(n-2)} H f - \tfrac{2n-3}{4(n-2)} i\, Hf \right\}.$$

From this expression of $(H^3, \bar{T}^2)'$ it is evident that $\bar{T}$ has $2(n-2)$ triple roots, namely the roots of $H = 0$; $3(n-2)$ double roots, namely the roots of $\bar{T} = 0$; n double roots, namely the roots of $f = 0$; $4(n-3)$ double roots, namely the roots of $(n-3) j f - (2n-3) i H = 0$.

The values of J giving the branch points of the Riemann surface may now be determined. Instead of $\bar{T}$ use the two equations

$$T(a) \cdot J f^2 (3 i H - 2 j f) = 6 H^3$$

$$T(b) \cdot (J-1) f^2 (3 i H - 2 j f) = -12 \bar{T}^2.$$

Then if $H = 0$, $J = 0$,

if $\bar{T} = 0$, $J = 1$,

if $f = 0$, $J = \infty$,

if $(n-3) j f - (2n-3) i H = 0$,

$$J = \frac{6(3-n) H^2}{(n+3) i f^3} = \frac{4 (3-n)^3 j^2}{(n+3)(2n-3)^2 i^3}.$$

If the $6(n-1)$ sheeted Riemann surface is regular, all the sheets, but at most, must come together over $J=\infty$. Assume that f and H have no common roots and that f has no double roots. Then the form of $(H, \bar{f}^{-1})$. It is evident that f accounts for $2n$ sheets. Hence there $6(n-1)-2n = 4(n-3)$ sheets to be brought together over $J=\infty$ for the roots of

$$(n-3)jf - (2n-3)iH = 0.$$

But $J=\infty$ only if $i=0$ or if $f=0$.

Let the branch points of $J=\infty$, other than those given by $f=0$, be determined by $\sqrt[k]{\ }$, where $k-1$ is the order of the branch point. Then, since the surface is regular,

$3(H-2)f$ is prefixed to $\sqrt[k-2]{\ }$ and $\sqrt[k]{\ }$

$(n-3)jf - (2n-3)iH$ " " " $\sqrt[k-2]{\ }$ or $\sqrt[k]{\ }$ where j refers to further branch points since $f=0$ to determine $J=\infty$. i is prefixed to $\sqrt[k]{\ }$. Consider the order of the quantities on both sides of the last proportionality. Let $\sqrt[k]{\ }$ indicate the order of the form $\sqrt[k]{\ }$. Then $4(n-3) = n(k-2) + k\sqrt[k]{\ }$. If

$k=2$, $\sqrt[2]{\ } = 2(n-3)$.

$\therefore$ f cannot be a factor of i, the degree of each term is $\geq 2(k-4)$ unless $i \equiv 0$.

If $k = 3$, then since H cannot contain f as a factor i contains f as a factor. But now $f = x - 4$. Hence

$$2(n-4) \geq k + n - 4 + h'$$ which is impossible

In fact since i H contains a factor f^{k-2} i contains a factor f^{k-2}. Hence

$$2(n-4) - f + n(-k-2) + h'$$

If $k > 3$, $h' + f \geq -8$. $\therefore$ $i \equiv 0$.

If $i \equiv 0$, $J = \infty$ is given by the roots of $f^3 = 0$ and the fourth points by f^2. Hence J is some power of f or

$$3(n-4) = \ell n$$

$$(3-\ell)n = 12$$

If $\ell = 0, 1, 2$ respectively, then $n = 4, 6, 12$ respectively. In these particular cases $i \equiv 0$. They are the well known tetrahedral, octahedral and icosahedral forms

Chapter II.

The Elliptic Normal Curve of Third Order, Δ_3, in Two-Dimensional Space, S_2.

§1. Introductory Remarks.

Since the general configuration of the elliptic normal curve of third order, Δ_3, in two-dimensional space is quite complicated, it is unnecessary to discuss it here, as it will offer an opportunity of giving a geometric significance to some of the algebraic results obtained in chapter I. Therefore only a few remarks on this subject will be given before the consideration of the elliptic modular function that is connected with Δ_3.

§2. The curve Δ_3.

According to the work of Klein and Brambilla, Δ_3, where three homogeneous coordi-

may be expressed as products of sigma functions, is transformed into itself by the group, G_{168} of collineations which sends the surface S_2 into itself. (vide K. F. II, p. 246-256). The normal form of its equation is shown to be

(1): $x_0^3 + x_1^3 + x_2^3 + 6\tau x_0 x_1 x_2 = 0$,

The reference triangle of the coordinate system in this case an inflexional Triangle. Since τ is defined by (1) in terms of a ratio of the x_i's, τ must itself be an elliptic modular function, i.e. a function of $u = \frac{\omega_1}{\omega_2}$ where ω_1 and ω_2 are the periods of the sigma functions in terms of which x_i is expressed. (vide §3, section 1).

§ 3. Discussion of τ.

If in (1) τ is considered as a parameter,

Two (1) represents a pencil of cubics on the nine inflexions. The degenerate cubics of the pencil are the inflexional Triangles. Therefore the proper cubics and the degenerate one intersect at the inflexions of the proper cubics.

Now an involution may be set up in the following fashion. Take a pencil of lines, involving a parameter t, on a common point of the pencil, i.e. on an inflexion. Every line of the pencil will cut the cubic in two more points, represented by a radical involving t and P. Since this radical is necessary to the discussion of P it will now be derived.

§4. The Involution determined by a Pencil of lines on an inflexion.

One inflexion of Δ_3 is the point $(0,1,1)$

a general of lines in this point is given by

(2): $x_2 = t(x_1 + x_2)$.

Substituting in (1) the values of $\frac{x_1}{x_2}$ obtained from (2'). Then it is found that

(3): $\frac{x_1}{x_2} = \frac{-2t^3 + 1 - 6\sqrt{2}t \pm \sqrt{3(8\sqrt{2}t^4 - 4t^3 + 12\sqrt{2}t^2 - 4\sqrt{2}t - 1)}}{2(t^3 + 1)}$

(4): Let $8\sqrt{2}t^4 - 4t^3 + 12\sqrt{2}t^2 - 4\sqrt{2}t - 1 \equiv \Delta_t^4$.

The values of t for which Δ_t^4 vanishes will, if substituted in (2), give the tangents from the point $(0, 1, -1)$, since in that case $\frac{x_1}{x_2}$ has no longer two values distinct. But no such tangent is the flex tangent at the point $(0, 1, -1)$. Since its parameter $t = \frac{1}{2\sqrt{2}}$, $t - \frac{1}{2\sqrt{2}}$ must be a factor of (4). In fact (4) may be written

(4'): $2\sqrt{2}(t - \frac{1}{2\sqrt{2}})(4t^3 + 6\sqrt{2}t + 1)$

Consider now the four flex lines on the point $(0, 1, -1)$. They are given by

$$(5)\quad \begin{cases} x_0 = c \\ x_0 + x_1 + x_2 = c \\ x_0 + \varepsilon(x_1 + x_2) = c \\ x_0 + \varepsilon^2(x_1 + x_2) = c \end{cases}$$

where ε equals a cube root of unity. The corresponding values of t are $1, -1, -\varepsilon, -\varepsilon^2$. The quartic with these values as roots is

$$(6):\ f_t^4 \equiv t_1^4 + t_1 t_2^3 = c$$

where the homogeneous variables t_1 and t_2 are used instead of t. If f_t^4 is rearranged in powers of $t = \frac{1}{2\tau}$ then f_t^4 (4') may be written

$$(7):\ 2\tau\left(t - \frac{1}{2\tau}\right) f_{\frac{1}{2\tau}}\, f_t^3 = c \quad \text{if } \tau' = \frac{1}{2\tau}$$

$$(7'):\ \frac{1}{k'}(t\tau')\, f_{\tau'}\, f_t^3$$

where $f_{\tau'} f_t^3$ indicates the polar of f_t^4 with respect to τ'. One may say then that, for a point of reference, the binary values, giving the parame-

two of the three tangents passed, other than the first tangent, to the point polar of the given quadric, using the parameters of the first lines on Δ. If the polar is taken with respect to the parameter of the first tangent at Δ.

30. For the Tetrahedron of Reciprocity.

If we

(4') $2\lambda(t-\frac{1}{2}\lambda)(4t^3-6\lambda t+1)$

the values $\tau_1 = t_1'$, $\tau_2 = 2\lambda t_1' - t_2'$ are introduced, then, (4') becomes

(5) $t_2'[4t_1'^3(1+8\lambda^3) - 36t_1'^2 t_2' + 12\lambda t_1' t_2^2 - t_2'^3]$.

$g_2 = 12\lambda(\lambda^3-1)$

$g_3 = 1 - 20\lambda^3 - 8\lambda^6$

$\Delta = -27(8\lambda^3+1)^3$

Then, in accordance with the Theory of Chapter [illegible] becomes in this case

(4) $T : T-1 : 1 = 64\tau(\tau^3+1)^3 : (1-20\tau^3-8\tau^6)^2 : (8\tau^3+1)^3$,

a form of the tetrahedral equation. (vide K. f. I, p. 243). If in (4) the values τ_1 τ_1, $\tau_2 = -2\tau_1$ then (9) becomes

(10) $J : J-1 : 1 = \tau^3(\tau^3+8)^3 : (\tau^6-20\tau^3-8)^2 : 64(\tau^3-1)^3$

the form of the modular equation given by Klein (see K. f. I, p. 104). In either case τ is the tetrahedron of irrationality. (vide B. for further discussion of τ).

Chapter III.

The Elliptic Normal Curve of Fourth Order, Ω_4, in Three-Dimensional Space, S_3.

§1.* Definition of Ω_4.

The coordinates of a point on the elliptic normal curve of fourth order, Ω_4, in three-dimensional space, S_3, are given by

$$x_\alpha(u) = c_\alpha \prod_{\mu=0}^{3} \sigma\left(u + \frac{\mu+1}{4}, \frac{\mu+2}{4}\right)(u \mid \omega_1, \omega_2),$$

where $\alpha = 0, 1, 2, 3$; c_α is an undetermined factor; and

$$\sigma\left(\frac{\alpha}{4}, \frac{\mu}{4}\right)(u \mid \omega_1, \omega_2) = e^{\left(\frac{\alpha\omega_1+\mu\omega_2}{4}\right)\left(u - \frac{\alpha\omega_1+\mu\omega_2}{8}\right)} \sigma\left(u - \frac{\alpha\omega_1+\mu\omega_2}{4} \mid \omega_1, \omega_2\right).$$

In order to obtain a convenient expression for c_α, Klein introduces the transformed periods $\bar{\omega}_1$, $\bar{\omega}_2$ such that

$$\bar{\omega}_1 = \omega_1, \quad \bar{\omega}_2 = \frac{\omega_2}{4}.$$

Because of the Legendre Relation $\omega_1 \eta_1 - \omega_2 \eta_2 = 2\pi i$

---

* Note: §1 and §2 do little more than give the theory stated by Klein (see K. F. III, pp. 261-265) for any Ω_n. In this case $n = 4$.

The corresponding substitution for η is

$$\bar{\eta}_1 = 4\eta_1, \quad \bar{\eta}_2 = \eta_2.$$

The coordinates are formally given in the form.

$$\chi_\lambda(u \mid \omega_1, \omega_2) = e^{-\left(\frac{\lambda+3}{4}\bar{\eta}_1 + \frac{\bar{\eta}_2}{2}\right)\left(\frac{\lambda+3}{4}\omega_1 + \frac{\omega_2}{2}\right)} \cdot e^{-\left(\frac{\bar{\eta}_1}{2\omega_1} - \frac{4}{\omega_1}\eta_1\right)u^2 + \left(\frac{\lambda+3}{4}\bar{\eta}_1 + \frac{\bar{\eta}_2}{2}\right)u}$$

$$\cdot \sigma\left(u - \frac{\lambda+3}{4}\omega_1 - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}\right),$$

where the notation indicates that $\chi_\lambda(u)$ is determined upon the periods ω_1, ω_2, and the sigma functions upon the periods $\omega_1, \frac{\omega_2}{4}$. In the formula $\omega_2 = e^{-\frac{1}{2}\left(\frac{\pi i}{4} + \frac{\eta_2 i}{2}\right)}$. $K \cdot K'$ indicates a prime term of u, independent of the periods. Let $e^{(\lambda)}$ represent the exponential factor involving the periods, which occurs in the formula given above. Then the homogeneous coordinates of a point in Δ_4 assume the following form:

$$
\begin{cases}
\chi_0(u) = a_0 e^{(0)} \cdot \sigma(u - \frac{\omega_1}{2} - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{\omega_1}{2} - \frac{3\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\qquad \cdot \sigma(u - \frac{\omega_1}{2} - \frac{5\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{\omega_1}{2} - \frac{7\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\chi_1(u) = a_1 e^{(1)} \cdot \sigma(u - \frac{3\omega_1}{4} - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{3\omega_1}{4} - \frac{3\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\qquad \cdot \sigma(u - \frac{3\omega_1}{4} - \frac{5\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{3\omega_1}{4} - \frac{7\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\chi_2(u) = a_2 e^{(2)} \cdot \sigma(u - \omega_1 - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \omega_1 - \frac{3\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\qquad \cdot \sigma(u - \omega_1 - \frac{5\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \omega_1 - \frac{7\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\chi_3(u) = a_3 e^{(3)} \cdot \sigma(u - \frac{5\omega_1}{4} - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{5\omega_1}{4} - \frac{3\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \\
\qquad \cdot \sigma(u - \frac{5\omega_1}{4} - \frac{5\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{5\omega_1}{4} - \frac{7\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4})
\end{cases}
\tag{11}
$$

The last coordinate may also be written

$$
\chi_5(u) = a_5 e^{(5) - \eta_1(4u + 2\omega_1 - 2\omega_2)} \cdot \sigma(u - \frac{\omega_1}{4} - \frac{\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{\omega_1}{4} - \frac{3\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{\omega_1}{4} - \frac{5\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}) \cdot \sigma(u - \frac{\omega_1}{4} - \frac{7\omega_2}{8} \mid \omega_1, \frac{\omega_2}{4}).
$$

In this coordinate system $\chi_{\lambda+4}(u) = \chi_\lambda(u)$.

Since the sum of the zeros of the eight functions occurring in any one coordinate is congruent to zero (mod. ω_1, ω_2), the condition that four points of the curve ξ_4 lie in a plane is

$$
u_1 + u_2 + u_3 + u_4 \equiv 0 \pmod{\omega_1, \omega_2}, \tag{12}
$$

where u_κ $(\kappa = 1, 2, 3, 4)$ are the arguments of the four points

Now if a singular point is defined as one in which a plane cuts Σ_4 in four consecutive points i.e. has three-fold contact with Σ_4, from (2) it is evident that Σ_4 has sixteen singular points whose arguments are $\frac{\mu\omega_1 + \nu\omega_2}{4}$ $(\mu, \nu = 0, 1, 2, 3)$. They may be arranged so that it is clear that they lie by fours on four planes.

$$(3)\quad \begin{cases} 0, & \frac{\omega_1}{2}, & \frac{\omega_2}{2}, & \frac{\omega_1+\omega_2}{2}, \\ \frac{\omega_1}{4}, & \frac{3\omega_1}{4}, & \frac{\omega_1+2\omega_2}{4}, & \frac{3\omega_1+3\omega_2}{4}, \\ \frac{\omega_2}{4}, & \frac{2\omega_1+\omega_2}{4}, & \frac{3\omega_1}{4}, & \frac{2\omega_1+3\omega_2}{4}, \\ \frac{\omega_1+\omega_2}{4}, & \frac{3\omega_1+\omega_2}{4}, & \frac{\omega_1+3\omega_2}{4}, & \frac{3\omega_1+3\omega_2}{4}. \end{cases}$$

In (3) the arguments of the singular points lying in one plane are given on one line. Therefore the singular points of Σ_4 lie by fours on the faces of a tetrahedron. In the case of Σ_3 (vide B) the nine singular points of the cubic lie by threes on the sides of four triangles. In the case of Σ_4 there is only one analogous tetrahedron, the one indicated by (3).

§ 2. The 32 collineations of the Ω_4 into itself.

The curve Ω_4 is transformed into itself by the group $(\rho,\sigma)_{2.4^2}$ of 32 collineations corresponding to the substitutions

$$(4) \qquad u' = \pm u + \frac{\rho\omega_1 + \sigma\omega_2}{4}, \quad (\rho,\sigma = 0,1,2,3).$$

The collineations thus obtained are those which transform Σ_2 into itself. They are of such a character that χ_2 and χ_3, connected with the elliptic integral of the first kind, do not vanish.

If all the substitutions included in (4) are necessary to consider only the three generating ones

$$(5). \qquad u' = -u, \quad u' = u - \frac{\omega_1}{4}, \quad u' = u - \frac{\omega_2}{4},$$

The effect of these substitutions upon the coordinates of a point of Ω_4 is as follows.

k_1	k_2	k_3
$\chi_0(-u) = \chi_0(u)$	$\chi_0(u-\frac{\omega_1}{4}) = \chi_1(u)$	$\chi_0(u-\frac{\omega_2}{4}) = \chi_0(u)$
$\chi_1(-u) = \chi_1(u)$	$\chi_1(u-\frac{\omega_1}{4}) = \chi_2(u)$	$\chi_1(u-\frac{\omega_2}{4}) = -\chi_1(u)$
$\chi_2(-u) = \chi_2(u)$	$\chi_2(u-\frac{\omega_1}{4}) = \chi_3(u)$	$\chi_2(u-\frac{\omega_2}{4}) = -\chi_3(u)$
$\chi_3(-u) = \chi_3(u)$	$\chi_3(u-\frac{\omega_1}{4}) = \chi_0(u)$	$\chi_3(u-\frac{\omega_2}{4}) = -\chi_1(u)$

If the right-hand side of K_2 is multiplied by $e^{-\pi_1(u-\frac{\omega_1}{2})}$ and the right hand side of K_3 by $e^{-\pi_2(u-\frac{\omega_2}{2})}$

Then K_1, K_2, K_3 have the same forms as those given by Klein under K. F. II, p 264). The fact of perfect reciprocity common to every number of any one collineation will always be omitted unless it is distinctly stated that reciprocation is present. This is permissible so long as the discussion actually involves only the nature of K_2.

The group $\gamma_{2.4^2}$ is defined essentially when the following relations between the generating elements K_1, K_2, K_3 are given.

$$K_1^2 = K_2^4 \quad K_3^4 = I, \quad K_2 K_3 = K_3 K_2,$$
$$K_1 K_2 = K_2^3 K_1, \quad K_1 K_3 = K_3^3 K_1.$$

Since it is easily seen that when K_1, K_2, K_3 have the forms given above, they do satisfy these relations, it is evident that K_1, K_2, K_3 generate a γ_{32} of collineations

In $\mathcal{G}_{32}$ there is one subgroup of order sixteen, $\mathcal{G}_{16}$, formed by K_2 and K_3. $\mathcal{G}_{16}$ itself is a self-conjugate subgroup under $\mathcal{G}_{32}$. Its elements form six cyclic invariant subgroups, $\mathcal{G}_4'$, each of which contains only one element of period two. Since this element of period two is the same for two $\mathcal{G}_4'$, there are only three such elements in $\mathcal{G}_{16}$. These three elements of period two together with the identity form a four group.

The remaining elements of $\mathcal{G}_{32}$ are obtained by adding K_1 to $\mathcal{G}_{16}'$. These obtained elements which are of period two, do not form a sub-group of $\mathcal{G}_{32}$. They may however be added to the cyclic $\mathcal{G}_4'$'s and the four-group so that $\mathcal{G}_{12}$ under $\mathcal{G}_{32}$ are obtained. In fact one cyclic $\mathcal{G}_4$ and one the four-group of order four is the most that may be formed.

Since the genetic significance of $\mathcal{G}_{32}$ and its subgroups can be discussed in a more

elegant form if a transformation of coordinates is made, the next paragraph will be concerned with the transformation which has proved to be a convenient one.

§3. A Transformation of Coordinates.

The elements of the form-group, R, are I and the only three elements of period two in $\frac{1}{6}$ (see §2 of this chapter). These elements are

K_2^2	K_3^2	$K_2^2 K_3^2$
$\chi_0(u-\frac{\omega_1}{2}) = \chi_2(u)$	$\chi_0(u-\frac{\omega_2}{2}) = \chi_0(u)$	$\chi_0(u-\frac{\omega_1+\omega_2}{2}) = \chi_2(u)$
$\chi_1(u-\frac{\omega_1}{2}) = \chi_3(u)$	$\chi_1(u-\frac{\omega_2}{2}) = -\chi_1(u)$	$\chi_1(u-\frac{\omega_1+\omega_2}{2}) = -\chi_3(u)$
$\chi_2(u-\frac{\omega_1}{2}) = \chi_0(u)$	$\chi_2(u-\frac{\omega_2}{2}) = \chi_2(u)$	$\chi_2(u-\frac{\omega_1+\omega_2}{2}) = \chi_0(u)$
$\chi_3(u-\frac{\omega_1}{2}) = \chi_1(u)$	$\chi_3(u-\frac{\omega_2}{2}) = -\chi_3(u)$	$\chi_3(u-\frac{\omega_1+\omega_2}{2}) = -\chi_1(u)$

Each of these collineations is an harmonic perspectivity in a fixed skew line from the equations of these lines are given in the table following:

for

K_2^2 : K_3^2 : $K_2^2 K_3^2$:

$L_1' \begin{cases} y_0 + y_2 = 0 \\ y_1 + y_3 = 0 \end{cases}$ $L_2' \begin{cases} y_0 = 0 \\ y_2 = 0 \end{cases}$ $L \begin{cases} y_0 + y_2 = 0 \\ y_1 - y_3 = 0 \end{cases}$

$L_1'' \begin{cases} y_0 - y_2 = 0 \\ y_1 - y_3 = 0 \end{cases}$ $L_2' \begin{cases} y_1 = 0 \\ y_3 = 0 \end{cases}$ $L_3'' \begin{cases} y_0 - y_2 = 0 \\ y_1 + y_3 = 0 \end{cases}$.

These fixed lines of Φ are also fixed under K_1 also.

Now consider the intersections of L_1.

L_1' and L_2' intersect in a point α, $(0, -1, 0, 1)$
L_1' and L_2'' intersect in a point β, $(-1, 0, 1, 0)$
L_1'' and L_2' intersect in a point γ, $(0, 1, 0, 1)$
L_1'' and L_2'' intersect in a point δ, $(1, 0, 1, 0)$ } I.

It is now easily seen that $\alpha, \beta, \gamma, \delta$ are the vertices of a tetrahedron T whose edges are fixed under the subgroup $\{A\}$ and the collineation, K_1.

The plane opposite α is $y_1 - y_3 = 0$.
The plane opposite β is $y_0 - y_2 = 0$.
The plane opposite γ is $y_1 + y_3 = 0$. } (a)

The plane of points x is $x_0 + x_2 = 0$ (1)

Let

$$T \begin{cases} y_0 = x_0 + x_2 \\ y_1 = x_1 + x_3 \\ y_2 = x_0 - x_2 \\ y_3 = x_1 - x_3 \end{cases}$$

Then α becomes $(0, 0, 0, 1)$,

β becomes $(0, 0, 1, 0)$,

γ becomes $(0, 1, 0, 0)$,

δ becomes $(1, 0, 0, 0)$.

T is now the reference tetrahedron. It is natural to choose T rather than to choose a tetrahedron fixed under a cycle S_4. T is connected with the only three elements of period two in S_6.

If the transformation (1) is applied to K_1, K_1, and K_3, these collineations assume the following forms:

C_1	C_2	C_3
$\eta_0(u) = \eta_0(u)$	$\eta_0(u-\frac{\omega_1}{4}) = \eta_1(u)$	$\eta_0(u-\frac{\omega_2}{4}) = \eta_2(u)$
$\eta_1(-u) = \eta_1(u)$	$\eta_1(u-\frac{\omega_1}{4}) = \eta_0(u)$	$\eta_1(u-\frac{\omega_2}{4}) = \eta_3(u)$
$\eta_2(-u) = \eta_2(u)$	$\eta_2(u-\frac{\omega_1}{4}) = \eta_3(u)$	$\eta_2(u-\frac{\omega_2}{4}) = \eta_0(u)$
$\eta_3(-u) = -\eta_3(u)$	$\eta_3(u-\frac{\omega_1}{4}) = -\eta_2(u)$	$\eta_3(u-\frac{\omega_2}{4}) = -\eta_1(u)$

[Note on the order of multiplication of substitutions. The $C_j C_i$, when C_i and C_j are elements of a group, is used to indicate multiplication from left to right. The method of combining the two operations is that followed by Klein and Fricke and is perhaps most easily understood if illustrated

C_1 C_2

$\eta_0(u'') \equiv \eta_0(-u') = \eta_0(-u+\frac{\omega_1}{4}) = \eta_1(u)$

$\eta_1(u'') \equiv \eta_1(-u') = \eta_1(-u+\frac{\omega_1}{4}) = \eta_0(u)$

$\eta_2(u'') \equiv \eta_2(-u') = \eta_2(-u+\frac{\omega_1}{4}) = \eta_3(u)$

$\eta_3(u'') \equiv \eta_3(-u') = -\eta_3(-u+\frac{\omega_1}{4}) = \eta_2(u)$

Usually only the last two columns are written.]

§4. Surfaces on which D_4 lies.

It is well known that an elliptic quadric cone is the complete intersection of two quadrics, which can be expressed as the sum of squares. Therefore each quadric and hence the curve, D_4, is unaltered by changing the signs of the variables. But changes of sign are caused by the elements of A together with the elements obtained by adding K_1 to A. Since the edges of the tetrahedron T are fixed under A and K_1, they are fixed under any combination of the elements of A and K_1. Therefore T is the unique self-polar tetrahedron of the system of quadrics that intersect in D_4.

The four cones, that are known to lie in the system of quadrics just mentioned, have as vertices the vertices of the self-polar tetrahedron, which has just been proved to be T. The equations of the four cones may now be obtained in a rather simple fashion. If one of the vertices, e.g. $A(0,0,0,1)$ is given by

(8) $$a y_0^2 + b y_1^2 + c y_2^2 = 0 .$$

Operate on (8) with ω_2, ω_3 and $\omega_2\omega_3$ in turn. The results are:

$$by_0^2 + ay_1^2 + cy_3^2 = 0 \,, \tag{9}$$

$$cy_0^2 + ay_2^2 - by_3^2 = 0 \,, \tag{10}$$

$$cy_1^2 - by_2^2 + ay_3^2 = 0 \,. \tag{11}$$

Then (8) - (11) obviously represent four cones, each one of four vertex of T. The method of determining these ends indicates that they can at the most be interchanged by ω_2 and ω_3. They are evidently unaltered by ω_1. Hence under the transformations of ω_{32} they can be permuted only.

But in the system of quadrics under discussion the theorem that four cones and only four cones occur. In other words any one obtained by any combination of the operations (8) - (11) must be one of the required four cones. For put a condition on the parameters a, b, c. If for instance (8) is multiplied by $\frac{b}{a}$, (9) multiplied by $\frac{a}{a}$ and the results subtracted, one obtains

$$(b^2-a^2)y_1^2 + bcy_2^2 - acy_3^2 = 0 \,. \tag{12}$$

But since this is the equation of a cone having the same vertex as (11), the coefficients of (12) must be proportional

to the coefficients of (11), i.e.

$$\frac{b^2-a^2}{c} = \frac{bc}{-b} = \frac{ac}{a} \quad \text{or} \tag{13}$$

$$a^2 = b^2 + c^2. \tag{14}$$

The same condition is derived if other combinations of the equations (8)–(11) are chosen. In terms of a single parameter, (14) may be replaced by

$$a = \Gamma_1^2 + \Gamma_2^2, \quad b = \Gamma_1^2 - \Gamma_2^2, \quad c = 2\Gamma_1\Gamma_2. \tag{15}$$

Since Q_4 is determined as the intersection of any two quadrics in the system, (8) and (9) may be conveniently used for this purpose. If the values given by (15) are introduced then Q_4 is the intersection of

$$(\Gamma_1^2+\Gamma_2^2)y_0^2 + (\Gamma_1^2-\Gamma_2^2)y_1^2 + 2\lambda_1\lambda_2 y_2^2 = 0 \tag{8'}$$

and

$$(\Gamma_1^2-\Gamma_2^2)y_0^2 + (\Gamma_1^2+\Gamma_2^2)y_1^2 + 2\lambda_1\lambda_2 y_2^2 = 0. \tag{9'}$$

The presence of the parameter Γ in (8') and (9') shows that the collineations of y_{12} determine not a single quartic curve, but a single infinity of such curves, just as the collineations of y_{18} in space S_2 determine a single infinity of cubics, the syzygetic pencil (vide B. p. 234, 55). Since then (8') and (9') give, from the parametric point of

more, systems of ∞ determining the family of quartics. The surface obtained by eliminating τ is one upon which all the quartics lie. The result of the elimination is

(8) $$\gamma_0^4 - \gamma_1^4 - \gamma_2^4 + \gamma_3^4 = 0.$$

§5. Discussion of G_{16}. The Field Spaces of the cyclic Sub-groups.

Let $B_j, (j = 1, \cdots, 6)$ represent the cyclic subgroups, G_4's, of G_{16}, which were mentioned in §2. The elements of B_1 are identity, I, and

C_2	C_2^2	C_2^3
$\gamma_0(u - \frac{\omega_1}{4}) = \gamma_1(u)$;	$\gamma_0(u - \frac{\omega_1}{2}) = \gamma_0(u)$;	$\gamma_0(u - \frac{3\omega_1}{4}) = \gamma_1(u)$;
$\gamma_1(u - \frac{\omega_1}{4}) = \gamma_0(u)$;	$\gamma_1(u - \frac{\omega_1}{2}) = \gamma_1(u)$;	$\gamma_1(u - \frac{3\omega_1}{4}) = \gamma_0(u)$;
$\gamma_2(u - \frac{\omega_1}{4}) = \gamma_3(u)$;	$\gamma_2(u - \frac{\omega_1}{2}) = -\gamma_2(u)$;	$\gamma_2(u - \frac{3\omega_1}{4}) = -\gamma_3(u)$,
$\gamma_3(u - \frac{\omega_1}{4}) = -\gamma_2(u)$;	$\gamma_3(u - \frac{\omega_1}{2}) = -\gamma_3(u)$;	$\gamma_3(u - \frac{3\omega_1}{4}) = \gamma_2(u)$.

The fixed lines are given by the following sets of equations:

for C_2: for C_2^2: for C_2^3:

a_1 $\begin{cases} y_0+y_1=0 \\ y_2+iy_3=0 \end{cases}$ L_1' $\begin{cases} y_0=0 \\ y_1=0 \end{cases}$ c_1 $\begin{cases} y_0+y_1=0 \\ y_2-iy_3=0 \end{cases}$

b_1 $\begin{cases} y_0-y_1=0 \\ y_2-iy_3=0 \end{cases}$ L_1'' $\begin{cases} y_2=0 \\ y_3=0 \end{cases}$ d_1 $\begin{cases} y_0-y_1=0 \\ y_2+iy_3=0 \end{cases}$

These fixed lines are the edges of a tetrahedron T_1 whose vertices are the intersections of three lines. The intersection of

a_1 and L_1' is the point $\alpha_1, (0,0,i,-1)$;
a_1 and L_1'' is the point $\beta_1, (-1,1,0,0)$;
b_1 and L_1' is the point $\gamma_1, (0,0,i,1)$;
b_1 and L_1'' is the point $\delta_1, (1,1,0,0)$.

} T_1.

The faces opposite

α_1 is given by $y_2-iy_3=0$
β_1 is given by $y_0-y_1=0$
γ_1 is given by $y_2+iy_3=0$
δ_1 is given by $y_0+y_1=0$

} (17).

The elements of B_2 are $1, C_3, C_3^2, C_3^3$. The fixed lines are given by the following sets of equations :-

for C_3: for C_3^2: for C_3^3:

a_2 $\begin{cases} y_0 + y_2 = 0 \\ y_1 + y_3 = 0 \end{cases}$ L_2' $\begin{cases} y_0 = 0 \\ y_2 = 0 \end{cases}$ c_2 $\begin{cases} y_0 + y_2 = 0 \\ y_1 - y_3 = 0 \end{cases}$

b_2 $\begin{cases} y_0 - y_2 = 0 \\ y_1 - y_3 = 0 \end{cases}$ L_2'' $\begin{cases} y_1 = 0 \\ y_3 = 0 \end{cases}$ d_2 $\begin{cases} y_0 - y_2 = 0 \\ y_1 + y_3 = 0 \end{cases}$

The vertices then of T_2 are the intersections of these pairs lines.

The lines.	Point of intersection.	
a_2 and L_2'.	$\alpha_2, (0, -1, 0, 1)$	T_2
c_2 and L_2''.	$\beta_2, (-1, 0, 1, 0)$	
c_2 and L_2'.	$\gamma_2, (0, 1, 0, 1)$	
b_2 and L_2''.	$\delta_2, (1, 0, 1, 0)$	

Point.	Opposite plane.	
α_2	$y_1 - y_3 = 0$	(18)
β_2	$y_0 - y_2 = 0$	
γ_2	$y_1 + y_3 = 0$	
δ_2	$y_0 + y_2 = 0.$	

The elements of B_2 are $I, c_2 c_3, c_2^2 c_3^2, c_2^3 c_3^3$. The fixed points are given by the following sets of equations:-

for $c_2 c_3$: for $c_2^2 c_3^2$: for $c_2^3 c_3^3$.

a_3 $\begin{cases} y_0 + i\sqrt{i}\, y_3 = 0 \\ y_1 + i\sqrt{i}\, y_2 = 0 \end{cases}$ L_3' $\begin{cases} y_0 = 0 \\ y_3 = 0 \end{cases}$ b_3 $\begin{cases} y_0 + i\sqrt{i}\, y_3 = 0 \\ y_1 - i\sqrt{i}\, y_2 = 0 \end{cases}$

a_3 $\begin{cases} y_0 - i\sqrt{i}\, y_3 = 0 \\ y_1 - i\sqrt{i}\, y_2 = 0 \end{cases}$ L_3'' $\begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases}$ b_3 $\begin{cases} y_0 - i\sqrt{i}\, y_3 = 0 \\ y_1 + i\sqrt{i}\, y_2 = 0 \end{cases}$

The vertices of T_3 are the intersections of these fixed lines.

The lines	Point of intersection	Opposite plane
a_3 and L_3':	$\alpha_3 (0, -i\sqrt{i}, 1, 0)$	$y_1 - i\sqrt{i}\, y_2 = 0$
a_3 and L_3'':	$\beta_3 (-i\sqrt{i}, 0, 0, 1)$	$y_0 - i\sqrt{i}\, y_3 = 0$
b_3 and L_3':	$\gamma_3 (0, i\sqrt{i}, 1, 0)$	$y_1 + i\sqrt{i}\, y_2 = 0$
b_3 and L_3'':	$\delta_3 (i\sqrt{i}, 0, 0, 1)$	$y_0 + i\sqrt{i}\, y_3 = 0$

T_3 (19)

The elements of B_4 are $I, c_2 c_3, c_2^2 \; c_2^3 c_3^3$. The fixed points are given by the following sets of equations:-

for $c_2 c_3$: for c_2^2:- for $c_2^3 c_3^3$.

a_4 $\begin{cases} y_0 + i y_1 = 0 \\ y_2 + y_3 = 0 \end{cases}$ L_1' $\begin{cases} y_0 = 0 \\ y_1 = 0 \end{cases}$ c_4 $\begin{cases} y_0 + i y_1 = 0 \\ y_2 - y_3 = 0 \end{cases}$

b_4 $\begin{cases} y_0 - i y_1 = 0 \\ y_2 - y_3 = 0 \end{cases}$ L_1'' $\begin{cases} y_2 = 0 \\ y_3 = 0 \end{cases}$ d_4 $\begin{cases} y_0 - i y_1 = 0 \\ y_2 + y_3 = 0 \end{cases}$

The vertices of T_4 are the intersections of these fixed lines.

The lines.	Point of intersection	Opposite plane.
a_4 and L_1'	α_4 : $(0, c, -1, 1)$.	$y_2 - y_3 = 0$
a_4 and L_1''	β_4 : $(-c, 1, 0, 0)$.	$y_0 - c y_1 = 0$
b_4 and L_1'	γ_4 : $(0, 0, 1, 1)$.	$y_2 + y_3 = 0$
b_4 and L_1''	δ_4 : $(c, 1, 0, 0)$.	$y_0 + c y_1 = 0$

T_4 (20)

The elements of B_5 are 1, $C_2^2 C_3$, C_3^2, $C_2^2 C_3^3$. The fixed lines are given by the following sets of equations:—

for $L_2^2 L_3$:	for L_3^2:	for $L_2^2 L_3^3$:
a_5 $\begin{cases} y_0 + c y_2 = 0 \\ y_1 + c y_3 = 0 \end{cases}$	L_2' $\begin{cases} y_0 = 0 \\ y_2 = 0 \end{cases}$	c_5 $\begin{cases} y_0 + c y_2 = 0 \\ y_1 - c y_3 = 0 \end{cases}$
b_5 $\begin{cases} y_0 - c y_2 = 0 \\ y_1 - c y_3 = 0 \end{cases}$	L_2'' $\begin{cases} y_1 = 0 \\ y_3 = 0 \end{cases}$	d_5 $\begin{cases} y_0 - c y_2 = 0 \\ y_1 + c y_3 = 0 \end{cases}$

The vertices of T_5 are the intersections of these fixed lines.

The lines	Point of intersection.	Opposite plane.
a_5 and L_2'.	α_5 : $(0, -c, 0, 1)$	$y_1 - c y_3 = 0$
a_5 and L_2''.	β_5 : $(-c, 0, 1, 0)$	$y_0 - c y_2 = 0$
b_5 and L_2'.	γ_5 : $(0, c, 0, 1)$	$y_1 + c y_3 = 0$
b_5 and L_2''.	δ_5 : $(c, 0, 1, 0)$	$y_0 + c y_2 = 0$

T_5 (21)

The elements of B_6 are I, $C_2^3 C_3$, $C_2^2 C_3^2$, $C_2 C_3^3$. The fixed lines are given by the following sets of equations:—

for $C_2^3 C_3$:

$$a_0 \begin{cases} y_0 + \sqrt{i}\, y_1 = 0 \\ y_1 + \sqrt{i}\, y_2 = 0 \end{cases} \qquad b_0 \begin{cases} y_0 - \sqrt{i}\, y_3 = 0 \\ y_1 - \sqrt{i}\, y_2 = 0 \end{cases}$$

for $C_2^2 C_3^2$:

$$L_3' \begin{cases} y_0 = 0 \\ y_3 = 0 \end{cases} \qquad L_3'' \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases}$$

for $C_2 C_3^3$:

$$c_0 \begin{cases} y_0 + \sqrt{i}\, y_3 = 0 \\ y_1 - \sqrt{i}\, y_2 = 0 \end{cases} \qquad d_0 \begin{cases} y_0 - \sqrt{i}\, y_3 = 0 \\ y_1 - \sqrt{i}\, y_2 = 0 \end{cases}$$

The vertices of T_6 are the intersections of these fixed lines.

The lines.	Point of intersection.	Opposite plane.
c_0 and L_3'.	$\alpha_6, (0, -1\sqrt{i}, 1, 0)$	$y_1 - \sqrt{i}\, y_2 = 0$
a_0 and L_3'.	$\beta_6, (-\sqrt{i}, 0, 0, 1)$	$y_0 - \sqrt{i}\, y_3 = 0$
b_0 and L_3''.	$\gamma_6, (0, \sqrt{i}, 1, 0)$	$y_1 + \sqrt{i}\, y_2 = 0$
d_0 and L_3'.	$\delta_6, (\sqrt{i}, 0, 0, 1)$	$y_0 + \sqrt{i}\, y_3 = 0$

(22)

A study of the Tables which have just been completed brings out the relations existing between T_j and the self-polar tetrahedron, T. Each pair of skew edges of T appear as edges of T_j and T_{j+1}. On each pair of skew edges of T are the vertices of T_j and T_{j+3} $(j = 1, 2, 3)$. The vertices of T_j are harmonic

with the vertices of T. The arrangement can be indicated thus:

α_1 and γ_1, lying on L_1', are harmonic with β and δ.

β_1 and δ_1, " " L_1'', " " " γ " α.

The same statements are true for $\alpha_4, \beta_4, \gamma_4, \delta_4$.

α_2 and γ_2, lying on L_2', are harmonic with α and γ.

β_2 and δ_2, " " L_2'', " " " β " δ.

The same statements are true for $\alpha_5, \beta_5, \gamma_5, \delta_5$.

α_3 and γ_3, lying on L_3', are harmonic with β and γ.

β_3 and δ_3, " " L_3'', " " " α " δ.

The same statements are true for $\alpha_6, \beta_6, \gamma_6, \delta_6$.

By an application of C_1, C_2, C_3 to the tables $T_1 - T_6$ it is evident that the vertices of each T_j are either unaltered or permuted among themselves. A study of the permutations will however be postponed until later.

§6. General discussion of the sixteen collineations not in G_{16}.

The sixteen elements of G_{32} not in G_{16} form a self-

conjugate set under G_{31}: But since the product of any two of these elements equals an element of G_{16}, the set does not form a group.

Two types of collineations are represented in the set, Four of the collineations represent harmonic perspectivities in each of which a vertex and opposite plane of T are fixed. The twelve collineations remaining represent harmonic perspectivities in each of which two skew lines are fixed. An examination of the form of the collineations expressed in terms of y makes evident the construction.

The four of the first type are

(23) C_1 fixed with $\eta_3 = 0$, $y_3 = 0$ fixed point and plane.

(24) $C_1 C_2^2$, " $\eta_2 = 0$, $y_2 = 0$ " " ".

(25) $C_1 C_3^2$, " $\eta_1 = 0$, $y_1 = 0$ " " " ".

(26) $C_1 C_2^2 C_3^2$ " $\eta_0 = 0$, $y_0 = 0$ " " " ".

The twelve of the second type with three pairs of perspectivity are

(27) $C_1 C_2$: $A_1 \begin{cases} y_0 + y_1 = 0 \\ y_2 + y_3 = 0 \end{cases}$ $B_1 \begin{cases} y_0 - y_1 = 0 \\ y_2 - y_3 = 0 \end{cases}$

(28) $C_1C_2^3$: $C_1\begin{cases} y_0+y_1=0 \\ y_2-y_3=0 \end{cases}$ $D_1\begin{cases} y_0-y_1=0 \\ y_2+y_3=0 \end{cases}$

(29) C_1C_3: $A_2\begin{cases} y_0+y_2=0 \\ y_1+iy_3=0 \end{cases}$ $B_2\begin{cases} y_0-y_2=0 \\ y_1-iy_3=0 \end{cases}$

(30) $C_1C_3^3$: $C_2\begin{cases} y_0+y_2=0 \\ y_1-iy_3=0 \end{cases}$ $D_2\begin{cases} y_0-y_2=0 \\ y_1+iy_3=0 \end{cases}$

(31) $C_1C_2^3C_3$: $A_3\begin{cases} y_0+i\sqrt{i}\,y_3=0 \\ y_1+\sqrt{i}\,y_2=0 \end{cases}$ $B_3\begin{cases} y_0-i\sqrt{i}\,y_3=0 \\ y_1-\sqrt{i}\,y_2=0 \end{cases}$

(32) $C_1C_2C_3^3$: $C_3\begin{cases} y_0-i\sqrt{i}\,y_3=0 \\ y_1-\sqrt{i}\,y_2=0 \end{cases}$ $D_3\begin{cases} y_0-i\sqrt{i}\,y_3=0 \\ y_1+\sqrt{i}\,y_2=0 \end{cases}$

(33) $C_1C_2C_3^2$: $A_4\begin{cases} y_0+iy_1=0 \\ y_2+iy_3=0 \end{cases}$ $B_4\begin{cases} y_0-iy_1=0 \\ y_2-iy_3=0 \end{cases}$

(34) $C_1C_2^3C_3^2$: $C_4\begin{cases} y_0+iy_1=0 \\ y_2-iy_3=0 \end{cases}$ $D_4\begin{cases} y_0-iy_1=0 \\ y_2+iy_3=0 \end{cases}$

(35) $C_1C_2^2C_3^3$: $A_5\begin{cases} y_0+iy_2=0 \\ y_1+iy_3=0 \end{cases}$ $B_5\begin{cases} y_0-iy_2=0 \\ y_1-iy_3=0 \end{cases}$

(36) $C_1C_2^2C_3$: $C_5\begin{cases} y_0+iy_2=0 \\ y_1-y_3=0 \end{cases}$ $D_5\begin{cases} y_0-iy_2=0 \\ y_1+y_3=0 \end{cases}$

(57) $C_1 C_2^3 C_3^3$: $A_6 \begin{cases} y_0 + \sqrt{-1}\, y_3 = 0 \\ y_1 + \sqrt{-1}\, y_2 = 0 \end{cases}$ $B_6 \begin{cases} y_0 - \sqrt{-1}\, y_3 = 0 \\ y_1 + \sqrt{-1}\, y_2 = 0 \end{cases}$

(58) $C_1 C_2 C_3$: $C_6 \begin{cases} y_0 + \sqrt{-1}\, y_2 = 0 \\ y_1 - \sqrt{-1}\, y_3 = 0 \end{cases}$ $D_6 \begin{cases} y_0 - \sqrt{-1}\, y_3 = 0 \\ y_1 + \sqrt{-1}\, y_2 = 0 \end{cases}$

The twenty-four axes of perfectivity form six skew quadrilaterals whose vertices include all possible intersections of any two axes. These vertices are found to be the same as the vertices of T_j and may be tabulated as follows. The first column gives two intersecting lines; the second-third point of intersection

A_1, C_1 : β_1	B_1, C_1 : γ_4
A_1, D_1 : α_4	B_1, D_1 : δ_1
A_2, C_2 : β_2	B_2, C_2 : γ_5
A_2, D_2 : α_5	B_2, D_2 : δ_2
A_3, C_3 : β_3	B_3, C_3 : γ_6
A_3, D_3 : α_6	B_3, D_3 : δ_3
A_4, C_4 : β_4	B_4, C_4 : γ_1
A_4, D_4 : α_1	B_4, D_4 : δ_4
A_5, C_5 : β_5	B_5, C_5 : γ_2

H_5, D_5 : α_2	B_5, D_5 : α_5
H_6, C_6 : β_6	B_6, C_6 : γ_3
H_6, D_6 : α_3	B_6, D_6 : δ_4

From the Table just given it is evident that the four lines with the same subscript form a linear quadrilateral. By reference to § 5 of the Lecture it is easy to see that the vertices of any one quadrilateral break into pairs harmonic with the three vertices of T paired off in some fashion.

The final points of the collineations (23) – (38) are of interest, particularly if their arguments are considered. For this reason I prefer the transformations of u which give rise to the collineations (23) – (38) are used. An accented number will indicate the transformation of u that gives rise to the collineation of the same number, e.g. (23') means $u' = -u$. The arguments of the period functions

for (23') :	0	$\frac{\omega_1}{2}$	$\frac{\omega_2}{2}$	$\frac{\omega_1+\omega_2}{2}$
(24') :	$\frac{\omega_1}{4}$	$\frac{3\omega_1}{4}$	$\frac{\omega_1+2\omega_2}{4}$	$\frac{3\omega_1+2\omega_2}{4}$

$(25')$:	$\frac{\omega_2}{4}$	$\frac{2\omega_1+\omega_2}{4}$	$\frac{3\omega_2}{4}$	$\frac{2\omega_1+3\omega_2}{4}$
$(26')$:	$\frac{\omega_1+\omega_2}{4}$	$\frac{3\omega_1+\omega_2}{4}$	$\frac{\omega_1+3\omega_2}{4}$	$\frac{3\omega_1+3\omega_2}{4}$
$(27')$:	$\frac{\omega_1}{8}$	$\frac{5\omega_1}{8}$	$\frac{\omega_1+4\omega_2}{8}$	$\frac{5\omega_1+4\omega_2}{8}$
$(28')$:	$\frac{3\omega_1}{8}$	$\frac{7\omega_1}{8}$	$\frac{3\omega_1+4\omega_2}{8}$	$\frac{7\omega_1+4\omega_2}{8}$
$(24')$:	$\frac{\omega_2}{8}$	$\frac{4\omega_1+\omega_2}{8}$	$\frac{5\omega_2}{8}$	$\frac{4\omega_1+5\omega_2}{8}$
$(30')$:	$\frac{3\omega_2}{8}$	$\frac{4\omega_1+3\omega_2}{8}$	$\frac{7\omega_2}{8}$	$\frac{4\omega_1+7\omega_2}{8}$
$(31')$:	$\frac{3\omega_1+\omega_2}{8}$	$\frac{\omega_1+\omega_2}{8}$	$\frac{3\omega_1+5\omega_2}{8}$	$\frac{\omega_1+5\omega_2}{8}$
$(32')$:	$\frac{\omega_1+3\omega_2}{8}$	$\frac{5\omega_1+3\omega_2}{8}$	$\frac{\omega_1+7\omega_2}{8}$	$\frac{5\omega_1+7\omega_2}{8}$
$(33')$:	$\frac{\omega_1+2\omega_2}{8}$	$\frac{5\omega_1+2\omega_2}{8}$	$\frac{\omega_1+6\omega_2}{8}$	$\frac{5\omega_1+6\omega_2}{8}$
$(34')$:	$\frac{3\omega_1+2\omega_2}{8}$	$\frac{7\omega_1+2\omega_2}{8}$	$\frac{3\omega_1+6\omega_2}{8}$	$\frac{7\omega_1+6\omega_2}{8}$
$(35')$:	$\frac{2\omega_1+3\omega_2}{8}$	$\frac{6\omega_1+3\omega_2}{8}$	$\frac{2\omega_1+7\omega_2}{8}$	$\frac{6\omega_1+7\omega_2}{8}$
$(36')$:	$\frac{2\omega_1+\omega_2}{8}$	$\frac{6\omega_1+\omega_2}{8}$	$\frac{2\omega_1+5\omega_2}{8}$	$\frac{6\omega_1+5\omega_2}{8}$
$(37')$:	$\frac{3\omega_1+3\omega_2}{8}$	$\frac{7\omega_1+3\omega_2}{8}$	$\frac{3\omega_1+7\omega_2}{8}$	$\frac{7\omega_1+7\omega_2}{8}$
$(38')$:	$\frac{\omega_1+\omega_2}{8}$	$\frac{5\omega_1+\omega_2}{8}$	$\frac{\omega_1+5\omega_2}{8}$	$\frac{5\omega_1+5\omega_2}{8}$

The arguments of the fixed points of (23') - (26') are the same as those of the singular points of Σ_4 (see (3), § 1 of this chapter). Moreover the fixed points of any one of the collineations (23) - (26) lie in a plane according to (2) § 1 Ch. III. But the fixed points of (23')

-(24) are the faces of the tetrahedron T. Therefore the singular points of Δ_4 lie by fours on the faces of the self-polar tetrahedron, T, of the system of quadrics intersecting in Δ_4.

§7. Degenerate curves in the Family of Quartics.

The curve, Δ_4, is the intersection of two whose equations (8) - (11) involve the parameters a, b, c, all of which may be expressed in terms of a single parameter λ. Accordingly

(15). $a = \lambda_1^2 + \lambda_2^2$, $b = \lambda_1^2 - \lambda_2^2$, $c = 2\lambda_1\lambda_2$.

If those values of λ for which a, b, c, satisfy, i.e. the roots of the octahedral cyclic $\lambda_1^5\lambda_2 - \lambda_1\lambda_2^5 = 0$, are substituted in the equations of the curve, the degenerate curves of the family of quartics are obtained. [illegible] each one degenerates into two [illegible] each one of T. The degenerate curves, which are therefore intersections of pairs, are found to be [illegible]

If $T = -1$, (3) and (4) become

(34) $\quad (y_0+y_1)(y_0-y_1) = 0\,;$

(9) and (10) become

(40) $\quad (y_2+iy_3)(y_2-iy_3) = 0.$

But the intersections of (34) and (40) are the lines a_1, b_1, c_1, d_1, i.e. edges of T_1. Similar results are obtained for the other roots of the octahedral cubic, and are given in the following table.

T	Coincident ones.	Degenerate curves.
0	$\begin{cases}(1)=(4)\\(10)=(11)\\(8)=(9)\end{cases}$	a_1, b_1, c_1, d_1, edges of T_1.
∞	$\begin{cases}(1)=(4)\\(10)=(11)\end{cases}$	a_4, b_4, c_4, d_4, " " T_4,
-1	$\begin{cases}(8)=(10)\\(9)=(11)\end{cases}$	a_2, b_2, c_2, d_2, " " T_2.
1	$\begin{cases}(8)=(10)\\(4)=(11)\end{cases}$	a_5, b_5, c_5, d_5, " " T_5.
$-i$	$\begin{cases}(8)=(11)\\(9)=(10)\end{cases}$	a_3, b_3, c_3, d_3, " " T_3,
i	$\begin{cases}(8)=(11)\\(9)=(10)\end{cases}$	a_6, b_6, c_6, d_6, " " T_6,

§ 8. Lines on the Surface, $y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0$.

The surface as a whole the family of quartic surfaces given by (16) § 4 ch. III.

(16). $y_0^4 - y_1^4 - y_2^4 + y_3^4 = 0.$

(16) may be written in two different forms:

(16'). $(y_0+y_1)(y_0-y_1)(y_0+iy_1)(y_0-iy_1)-(y_2+y_3)(y_2-y_3)(y_2+iy_3)(y_2-iy_3)=0$

(16''): $(y_0+y_2)(y_0-y_2)(y_0+iy_2)(y_0-iy_2)-(y_1+y_3)(y_1-y_3)(y_1+iy_3)(y_1-iy_3)=0$

(16'''): $(y_0+\sqrt{i}y_3)(y_0-\sqrt{i}y_3)(y_0+\sqrt{-i}y_3)(y_0-\sqrt{-i}y_3)-$

$(y_1+i\sqrt{i}y_2)(y_1-i\sqrt{i}y_2)(y_1+\sqrt{-i}y_2)(y_1-\sqrt{-i}y_2)=0.$

It is evident from the form of these equations that the intersection of a π-plane passing a factor of the first term of (16'), (16''), (16''') with a π-plane passing a factor of the second term of the same equation is a line lying on the surface (16). By referring to §5 and §6 Ch. II, it is seen that the forty-eight lines thus obtained from (16), (16'), (16'') are the twenty four lines which make up the degenerate curves of the family and the twenty-four axes of perspectivity of the twelve collineations (27)-(38).

If now the method of §§ 5, 6, 7 of this chapter is recalled, the configuration of the forty-eight lines — the

surface (16) becomes modernistically clear. The double points of the degenerate curves lie on the edges of T. Since the edges of T do not cut a proper D_4 - a fact easily proved algebraically - the double points of the degenerate curves do not lie on a proper D_4. In the case of the cubic, D_3, the opposite is true. In fact no intersection of any degenerate curve with either a degenerate or non-degenerate D_4 can be found. There are however intersections with the axes of perspectivity. On every double point are two axes. A comparison of the tables (nos. 5 and 8) giving the vertices of the above quadrilaterals, which are the present subject of discussion shows that the quadrilateral composed of axes of perspectivity has two vertices at the double points of one degenerate curve, two at the double points of another. The pair of vertices mentioned are on three edges of T. In addition to the intersections of each axis with two degenerate curves, at their double points, each axis cuts two lines of the remaining four de-

generate curves. A pair of axes associated with an collineation cut the curves two times of any one of these four degenerate curves. One axis of perspectivity intersects a proper D_4 twice. These situations will be considered later

§9. Behavior of the Skew Quadrilaterals under G_8's.

Consider first a degenerate curve, e.g. the one whose double points are the vertices of T_1. These vertices are (vide §5)

$$\left.\begin{array}{l} \alpha_1, (0, 0, \epsilon, -1) \\ \beta_1, (-1, 1, 0, 0) \\ \gamma_1, (0, 0, \epsilon, 1) \\ \delta_1, (1, 1, 0, 0) \end{array}\right\} T_1$$

Any power of C_2 leaves each vertex unaltered. The effect of C_3 is the same as that of the permutation $(\alpha_1 \beta_1 \gamma_1 \delta_1)$. Therefore any combination of C_2^a and C_3^b is the same as that of $(\alpha_1 \beta_1 \gamma_1 \delta_1)^k$ where obviously $k = 4$ gives the identical permutation.

The effect of C_1 is given by $(\alpha_1\beta_1)(\gamma_1\delta_1)$ which is evidently a permutation of powers of two. Any combination of C_1 and C_3^k effects the permutation obtained by using the corresponding permutations. If then H_1 indicates the tetrahedral group generated over the $\gamma_1\delta_1$ subgroup of B_1 by the addition of C_1, it is easily seen that the permutations, under H_2, of the sixes of the degenerate curve, that are edges of T_1 are simply a morphism with the tetrahedral permutation group represented by $(\alpha_1\beta_1\gamma_1\delta_1)^4 = (\alpha_1\beta_1)(\gamma_1\delta_1)^2 = 1$. H_2 includes among its elements of γ_{32} that effects the edges of T_1. The two edges of T_1 that are edges of T_2 it leaves unaltered. The four remaining edges of T_1 it permutes in such a way that members of a skew pair are interchanged. Similar results are obtained for H_2, and may be tabulated thus:

The vertices of T_1 are permuted by H_2.

" " " T_2 " " " H_1.

The vertices of T_5 are permuted by H_5

" " " T_4 " " " H_5.

" " " T_5 " " " H_6

" " " T_6 " " " H_4.

Under each H_i, the two edges of T and T_i have in common are unaltered, while the other four edges of T are permuted in such a way that members of a skew pair are interchanged.

Now let the general configuration of the four skew quadrilaterals whose vertices are on two skew edges of T be studied. Consider for instance the edges of T common to T_1 and T_4. Two degenerate curves a_1, b_1, c_1, d_1 and a_4, b_4, c_4, d_4 have their vertices on L_1' and L_1''. Two skew quadrilaterals composed of axes of perspectivities have the same vertices. These quadrilaterals are A_1, B_1, C_1, D_1 and A_4, B_4, C_4, D_4 (notation). H_2 and H_3 together have the effect either of interchanging the skew pairs of one quadrilateral

with a third pair of the same quadrilateral or of interchanging the two quadrilaterals. Similar results may be obtained if other edges of T are considered.

§10. Projection of Q_4 on a Plane of T.

The quartic curve, Q_4, is transformed into itself by G_1, an involutory perspectivity on the fixed point and plane $y_3 = 0$, $y_3 = 0$ of the tetrahedron T. In other words every point of Q_4 is united to another point of Q_4 both of which lie on a line through $y_3 = 0$. Therefore two points of Q_4 correspond to one point on the section of the perspectivity made by $y_3 = 0$. Since Q_4 lies on the cone with vertex at $y_3 = 0$, the section of which is really the projection of Q_4 on $y_3 = 0$, is obtained by setting $y_3 = 0$ in the equation (8) which represents the cone with vertex at $y_3 = 0$. The projection then of Q_4 on $y_3 = 0$ is a doubly covered conic:

(8) $a y_0^2 + b y_1^2 + c y_2^2 = 0$.

This may be represented parametrically

$$(41)\quad \begin{cases} y_0 = \sqrt{bc}\,(t_1^2 + t_2^2) \\ y_1 = i\sqrt{ca}\,(t_1^2 - t_2^2) \\ y_2 = i\sqrt{ab}\,(2 t_1 t_2) \end{cases}$$

Since two points that project into the same point on $y_3 = 0$ have only their fourth coordinates different, members of a pair of corresponding points can be separated by the values of their fourth coordinates. Now Σ_4 is determined as the intersection of two cones, e.g. (8) and (4). From (4)

$$(42)\quad y_3 = \pm\sqrt{\frac{-b y_0^2 - a y_1^2}{c}}.$$

Since any two corresponding points of Σ_4 are separated by the radical (42).

Introduce in (42) the parametric values of y_i given by (41) and then the values of a, b, c given in (15). The result is

$$(42')\quad y_3 = \pm 2\sqrt{T_1^2 T_2^2 (t_1^4 + t_2^4) - t_1^2 t_2^2 (T_1^4 + T_2^4)}.$$

If $f_t^6 \equiv t_1^5 t_2 - t_1 t_2^5$, an octahedral curve,

then (42') may be written

(42'') $$y_3 = \pm 2\sqrt{2(t\tau) f_\tau^3 f_t^3}$$

where $(t\tau) \equiv (t_1\tau_2 - t_2\tau_1)$ and $f_\tau^3 f_t^3$ is the Jacobian of f_t^6 taken with respect to τ.

In order that the plane $y_3 = 0$ cut Σ_4 in four singular points, then points must have for their parameters those values of t for which (42') vanishes. These values are

(43) $$\frac{t_1}{t_2} = \pm \frac{\tau_1}{\tau_2}, \quad \pm \frac{\tau_2}{\tau_1}$$ or

(43') $$t = \pm\tau, \pm\frac{1}{\tau}.$$

Hence the coordinates of the four singular points in the plane $y_3 = 0$ are

(44) $$\begin{cases} y_0 = \sqrt{a} \\ y_1 = i\sqrt{b} \\ y_2 = \pm i\sqrt{c} \\ y_3 = 0 \end{cases}$$ and (45) $$\begin{cases} y_0 = \sqrt{a} \\ y_1 = -i\sqrt{b} \\ y_2 = \pm i\sqrt{c} \\ y_3 = 0 \end{cases}$$

Evidently if a singular point on $y_3 = 0$ the corresponding points of Σ_4 have coincided with the point into which they have been projected

The line of projection, which is common a generator of cone (8), carries with it a plane of the cone (8), which in general touches Δ_4 at the two corresponding points lying on the generator. In the particular case then, when two corresponding points have coincided with their projection which is a point of Δ_4, the tangent plane of the cone touches Δ_4 in three coincident points at a singular point and must therefore because of the nature of a singular point have three-fold contact with Δ_4 at that point. Therefore the tangent plane of the cone is a stationary plane of the curve. It is evident then that each cone (8) - (11) has four tangent planes, which are stationary planes of Δ_4. The points of contact in each case are the singular points in the plane of T opposite the vertex of the cone in question (vide S. p. 301, ¶ 4).

The coordinates of the singular points not

$= y_4 = 0$ and determined of the order of $(t^2)^6 = 12$ as usual.
$t = 0, t = \infty$, give the singular points on $y_2 = 0$;
$t = \pm 1$, those on $y_1 = 0$; $t = \pm i$; those on $y_0 = 0$.
The results of the substitution of these values of t
together with $t = \pm \sqrt{}, \pm \frac{1}{\sqrt{}}$ are collected in the
following Table of the coördinates of the singular points.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
y_0:	0	0	0	0	$\sqrt{c}$	$\sqrt{c}$	$\sqrt{c}$	$\sqrt{c}$
y_1:	$i\sqrt{c}$	$i\sqrt{c}$	$i\sqrt{c}$	$i\sqrt{c}$	0	0	0	0
y_2:	$\sqrt{b}$	$\sqrt{b}$	$-\sqrt{b}$	$-\sqrt{b}$	$i\sqrt{a}$	$i\sqrt{a}$	$-i\sqrt{a}$	$-i\sqrt{a}$
y_3:	$-\sqrt{a}$	$\sqrt{a}$	$-\sqrt{a}$	$\sqrt{a}$	$i\sqrt{b}$	$-i\sqrt{b}$	$i\sqrt{b}$	$-i\sqrt{b}$

	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
y_0:	$\sqrt{b}$	$\sqrt{b}$	$\sqrt{b}$	$\sqrt{b}$	$\sqrt{a}$	$\sqrt{a}$	$\sqrt{a}$	$\sqrt{a}$
y_1:	$-i\sqrt{a}$	$-i\sqrt{a}$	$i\sqrt{a}$	$i\sqrt{a}$	$i\sqrt{b}$	$i\sqrt{b}$	$-i\sqrt{b}$	$-i\sqrt{b}$
y_2:	0	0	0	0	$i\sqrt{c}$	$-i\sqrt{c}$	$i\sqrt{c}$	$-i\sqrt{c}$
y_3:	$\sqrt{c}$	$-\sqrt{c}$	$\sqrt{c}$	$-\sqrt{c}$	0	0	0	0

$s_1 - s_{12}$ lie by fours in lines through $y_3 = 0$. Corresponding points are separated by the signs of y_3, e.g. s_1 and s_2 are on one line, s_3 and s_4 are on another etc.

In chapter II, § 4 a general statement concerning the binary form obtained by the projection of Γ_3 was made. An analogous but not coincident statement must be made for Γ_4.

(4c) The t parameters of the cuspidal tangents of the twelve singular points not in the plane of projection form an octahedral cycle $f_t^{\frac{t}{2}=0}$. The t parameters of the four singular points in the plane of projection are given by $(t\Gamma)^2_{\Gamma}(t^3)^2_{\Gamma}$ a form obtained by taking the third polar of f_t^6 with respect to Γ and multiplying the result by $(t\Gamma)$.

Similar results are obtained if Γ_4 is projected by any one of the four cases (1) - (11), (4c) is of general application, and is not true for only the case by which it was derived.

From the form of the coordinates of the singular points it is evident that none of them can be edges of T. Since each face of T cannot have

more than four intersections with Δ_4, which are already accounted for by the singular points, no portion of Δ_4 can lie on an edge of T.

The projection of Δ_4 on $\eta_3 = 0$ has seven cusps and twelve ordinary points only twice but Δ_4 can be projected within three other faces of T. Hence there are all together twenty-four lines on each of which are two singular points. On each singular point are three of these lines, one from each vertex of T lying in the plane of the singular point

In order to determine the arguments of the singular points which lie on the same line, it is only necessary to refer to the transformations (23') - (26') on (3). The character of the collineations (23) - (26) show that arguments three transformed into each other belong to corresponding points lying on a line through a vertex of T. If (23') is used, the singular

points on the plane $\eta_3 = 0$ are considered. The other lines are seen through $\eta_3 = 0$.

On the plane, the arguments of the singular points on

	first line		second line	
$\eta_0 = 0$	$\frac{w_1+w_2}{4}$, $\frac{3w_1+3w_2}{4}$	;	$\frac{3w_1+w_2}{4}$,	$\frac{w_1+3w_2}{4}$
$\eta_1 = 0$	$\frac{w_2}{4}$, $\frac{3w_2}{4}$	;	$\frac{2w_1+w_2}{4}$,	$\frac{2w_1+3w_2}{4}$
$\eta_2 = 0$	$\frac{w_1}{4}$, $\frac{3w_1}{4}$	;	$\frac{w_1+2w_2}{4}$,	$\frac{3w_1-2w_2}{4}$

(47)

If (24') is used, the singular points on the plane $\eta_2 = 0$ are considered. The others lie on lines through $\eta_2 = 0$.

On the plane, the arguments of the singular points on

	first line		second line	
$\eta_0 = 0$	$\frac{w_1+w_2}{4}$, $\frac{w_1+3w_2}{4}$	;	$\frac{3w_1+w_2}{4}$,	$\frac{3w_1+3w_2}{4}$
$\eta_1 = 0$	$\frac{w_2}{4}$, $\frac{2w_1+3w_2}{4}$	;	$\frac{2w_1+w_2}{4}$,	$\frac{3w_1}{4}$
$\eta_3 = 0$	0, $\frac{w_1}{2}$	;	$\frac{w_2}{2}$,	$\frac{w_1+w_2}{2}$

(48)

If (25') is used, the singular points on the plane $\eta_1 = 0$ are considered. The other lines lie on lines through $\eta_1 = 0$.

On the plane, the arguments of the singular points on

	first line		second line	
$\eta_0 = 0$	$\frac{w_1+w_2}{4}$, $\frac{w_1+3w_2}{4}$	;	$\frac{2w_1+w_2}{4}$,	$\frac{3w_1+3w_2}{4}$
$\eta_2 = 0$	$\frac{w_1}{4}$, $\frac{w_1+2w_2}{4}$	;	$\frac{3w_1}{4}$,	$\frac{w_1+3w_2}{4}$
$\eta_3 = 0$	0, $\frac{w_2}{2}$	;	$\frac{w_1}{2}$,	$\frac{w_1-w_2}{2}$

(49)

If (26') is used, the singular points on the plane $\gamma_0 = 0$ are marked out. The other lines are drawn through $\gamma_1 = 0$.

On the plane, the arguments of the singular points are:

	first line		second line	
$\gamma_1 = 0$	$\frac{\omega_2}{4}$,	$\frac{2\omega_1+\omega_2}{4}$;	$\frac{3\omega_2}{4}$,	$\frac{2\omega_1+3\omega_2}{4}$
$\gamma_2 = 0$	$\frac{\omega_1}{4}$,	$\frac{\omega_1+2\omega_2}{4}$;	$\frac{3\omega_1}{4}$,	$\frac{3\omega_1+2\omega_2}{4}$
$\gamma_3 = 0$	0,	$\frac{\omega_1+\omega_2}{2}$;	$\frac{\omega_1}{2}$,	$\frac{\omega_2}{2}$.

(50)

§11. The Parameter τ.

By (8') which is written

(8') $(\Gamma_1^2+\Gamma_2^2)\gamma_0^2 + (\Gamma_1^2-\Gamma_2^2)\gamma_1^2 + 2\Gamma_1\Gamma_2\gamma_2^2 = 0$

the parameter τ is determined as an elliptic modulus function, i.e. as function of $w = \frac{\omega_1}{\omega_2}$.

From (8') is obtained the value

(51) $$\tau = \frac{\Gamma_1}{\Gamma_2} = \frac{-\gamma_2^2 \pm \sqrt{\gamma_2^4 - \gamma_0^4 + \gamma_1^4}}{\gamma_0^2 + \gamma_1^2}.$$

Make use of (16) (with §4). Then

(52) $$\tau = \frac{-\gamma_2^2 \pm \gamma_3^2}{\gamma_0^2 + \gamma_1^2}$$

If τ is evaluated from the argument $\frac{\omega_1}{4}$ (51) the expression

$$(53)\quad \tau = \frac{-y_2^2\left(\frac{\omega_1}{4}\right) \pm y_3^2\left(\frac{\omega_1}{4}\right)}{y_0^2\left(\frac{\omega_1}{4}\right) + y_1^2\left(\frac{\omega_1}{4}\right)}.$$

Apply ζ_1 to (53).

$$(54)\quad \tau = \frac{-y_3^2(0) \pm y_2^2(0)}{y_1^2(0) + y_0^2(0)}.$$

Express y_i in terms of x_i by means of (7).

$$(55)\quad \tau = \frac{-[x_1(0)-x_3(0)]^2 \pm [x_2(0) - x_4(0)]^2}{[x_1(0)+x_3(0)]^2 + [x_0(0) + x_2(0)]^2}.$$

By substituting for x_i the values given by (1) for $u = 0$, an expression for τ in terms of sigma functions can be obtained.

Another information concerning τ can be obtained by a consideration of (42') and (42). In §10 it was noted that

$$(56)\quad T_1^2 T_2^2 (t_1^4 - t_2^4) - (T_1^4 + T_2^4) t_1^2 t_2^2 = (tT)\, f_T\, f_t^5,$$

where $f_T^5 f_t^5$ is the third polar of

$$(57)\quad f_t^6 \equiv t_1^5 t_2 - t_1 t_2^5.$$

The quantic in t given by (56) is therefore of the type discussed in Chapter II. Hence the invariants of this quantic, J_2, J_3, Δ can be ex-

formed in terms of the transcendents of f_τ^6.

Let $n = 4$ in § 4 chapter I. Then the forms there given are defined for the case now being considered. If now the results of chapter I are used, it is known that

$$g_2 = -\frac{3}{5} H, \quad g_3 = \frac{1}{16} \bar{t}, \quad \Delta = 2\bar{t}^2 + H^3 = \Delta \tag{58}$$

But for the modulus (56)

$$\begin{cases} g_2 = \dfrac{3(\lambda_1^8 + 14\lambda_1^4\lambda_2^4 + \lambda_2^8)}{64} \\ g_3 = (\lambda_1^{12} - 33\lambda_1^8\lambda_2^4 - 33\lambda_1^4\lambda_2^8 + \lambda_2^{12})\dfrac{1}{1024} \\ \Delta = -\dfrac{108}{5832}(\lambda_1^4 - \lambda_2^4)^4\lambda_1^4\lambda_2^4 . \end{cases} \tag{59}$$

Determine the values of $H, \bar{t}, \Delta'$ from (58) and (59) and introduce them in the equation V (vide § 7, I). The result is

$$\begin{aligned} J\,|\,J-1\,,\,1 = {} & (\lambda_1^8 + 14\lambda_1^4\lambda_2^4 + \lambda_2^8)^5 ; \\ & (\lambda_1^{12} - 33\lambda_1^8\lambda_2^4 - 33\lambda_1^4\lambda_2^8 + \lambda_2^{12})^2 ; \\ & 108\,\lambda_1^4\lambda_2^4(\lambda_1^4 - \lambda_2^4)^4 . \end{aligned} \tag{60}$$

But (60) is the octahedral equation (vide K.F. I, p. 210).

Therefore T as an algebraic function of T is the rational involutorial.

§12 Projection of Q_4 on an Axis of Perspectivity

It has already been noticed that each of the collineations (27) - (38) is an harmonic perspectivity on a skew line pair (vide §61). Since each axis cuts Q_4 in two points, there must be the fixed points of the collineation. Hence the four fixed points of each collineation (vide § 6, (27') - (38') lie by pairs on the two fixed lines of the same collineation. The intersections of the axes can be given in terms of the parameters by the following table.

	Q_4 is cut by the line,	in the points
A_1		$(\mp i\sqrt{c}, \pm i\sqrt{c}, -\sqrt{a+b}, \sqrt{a+b})$
B_1		$(\pm i\sqrt{c}, \pm i\sqrt{c}, \sqrt{a+b}, \sqrt{a+b})$
C_1		$(\mp i\sqrt{c}, \pm i\sqrt{c}, \sqrt{a+b}, \sqrt{a+b})$
D_1		$(\pm i\sqrt{c}, \pm i\sqrt{c}, -\sqrt{a+b}, \sqrt{a+b})$

H_2 $\quad (\mp\sqrt{b},\ -i\sqrt{a+c} \pm \sqrt{b},\ \sqrt{a+c})$

B_2 $\quad (\pm\sqrt{b},\ i\sqrt{a+c},\ \pm\sqrt{b},\ \sqrt{a+c})$

C_2 $\quad (\mp\sqrt{b},\ i\sqrt{a+c},\ \pm\sqrt{b},\ \sqrt{a+c})$

D_2 $\quad (\pm\sqrt{b},\ -i\sqrt{a+c},\ \pm\sqrt{b},\ \sqrt{a+c})$

H_3 $\quad (-\sqrt{b-ic},\ \mp i\sqrt{a},\ \pm\sqrt{ia},\ \sqrt{ib+c})$

B_3 $\quad (\sqrt{b-ic},\ \pm i\sqrt{a},\ \pm\sqrt{ia},\ \sqrt{ib+c})$

C_3 $\quad (-\sqrt{b-ic},\ \pm i\sqrt{a},\ \pm\sqrt{ia},\ \sqrt{ib+c})$

D_3 $\quad (\sqrt{b-ic},\ \mp i\sqrt{a},\ \pm\sqrt{ia},\ \sqrt{ib+c})$

H_4 $\quad (\mp i\sqrt{c},\ \pm\sqrt{c},\ -\sqrt{a-b},\ \sqrt{b-a})$

B_4 $\quad (\pm i\sqrt{c},\ \pm\sqrt{c},\ \sqrt{a-b},\ \sqrt{b-a})$

C_4 $\quad (\mp i\sqrt{c},\ \pm\sqrt{c},\ \sqrt{a-b},\ \sqrt{b-a})$

D_4 $\quad (\pm i\sqrt{c},\ \pm\sqrt{c},\ -\sqrt{a-b},\ \sqrt{b-a})$

H_5 $\quad (\mp i\sqrt{b},\ -\sqrt{a-c},\ \pm\sqrt{b},\ \sqrt{a-c})$

B_5 $\quad (\pm i\sqrt{b},\ \sqrt{a-c},\ \pm\sqrt{b},\ \sqrt{a-c})$

C_5 $\quad (\mp i\sqrt{b},\ \sqrt{a-c},\ \pm\sqrt{b},\ \sqrt{a-c})$

D_5 $\quad (\pm i\sqrt{b},\ -\sqrt{a-c},\ \pm\sqrt{b},\ \sqrt{a-c})$

H_6 $\quad (-\sqrt{b+ic},\ \mp\sqrt{a},\ \pm i\sqrt{ia},\ \sqrt{c-ib})$

B_6 $\quad (\sqrt{b+ic},\ \pm i\sqrt{a},\ \pm i\sqrt{ia},\ \sqrt{c-ib})$

C_6 $\quad (-\sqrt{b+ic},\ \pm i\sqrt{a},\ \pm i\sqrt{ia},\ \sqrt{c-ib})$

D_6 $\quad (\sqrt{b+ic},\ \mp i\sqrt{a},\ \pm i\sqrt{ia},\ \sqrt{c-ib})$

The upper signs of the radicals are to be taken together and the lower signs are to be treated similarly.

Since every axis cuts the quartic curve Q_4 in two points, any plane on an axis can cut Q_4 in only two other points. If then a pencil of planes on an axis is taken, an involution on Q_4 is established, and so that no point determines another. Furthermore since any one of the planes in the pencil cuts the other fixed axis of the collineation under consideration in only one point, a two-to-one correspondence between the points of the curve and the points of a line is established. In other words two points of Q_4 are projected into one point on an axis by the lines joining the two points of Q_4.

In order to determine the radical which separates two corresponding points of the involution, a particular collineation will be assumed, e.g. (24): $C_1 C_2$. A pencil of planes

H_1 is given by

(61) $\qquad y_0 + y_1 + d(y_2 + y_3) = 0.$

To find the intersection of the plane (61) and Δ_4 use the two equations (8) and (9):

(8) $\qquad a y_0^2 + b y_1^2 + c y_2^2 = 0$

(9) $\qquad a y_0^2 + a y_1^2 + c y_3^2 = 0.$

By eliminating $\frac{y_0}{y_3}$ and $\frac{y_1}{y_3}$ from (61), (8) and (9) a single equation for the determination of $\frac{y_2}{y_3}$ is obtained:

(62) $\quad d^2(y_2+y_3)^2(a^2-b^2)^2 + 2cd^2(y_2+y_3)^2(y_2^2+y_3^2)(a+b)(a-b)^2$
$\qquad + c^2(a+b)^2(y_2^2-y_3^2)^2 = 0$

(62), a quartic in $\frac{y_2}{y_3}$ is divisible by $(y_2+y_3)^2$ the two factors corresponding to the two points in which H_1 cuts Δ_4 (vide table at beginning of this paragraph). The quadratic remaining gives the coordinates of corresponding points of the involution. If this quadratic is solved for $\frac{y_2}{y_3}$ it is found that

(63) $\qquad \frac{y_2}{y_3} = \frac{(a+b)[c^2 - d^2(a-b)^2]}{d^2c^2(a-b)^2 + c d^2(a-b)^2 + c^2(a+b)}$

$$\frac{\pm 2\,c\,c\,|(a-b)|\,\delta^4 c^4(a-b)+2\delta^2(a^2+b^2)+c(a+b)}{\delta^4 c^2(a-b)+2c\delta^2(a-b)^2+c^2(a+b)}$$

The quantity under the radical in the expression corresponding to (64) can be written homogeneously in terms of $\tau_1, \tau_2, \delta_1, \delta_2$ where $\delta = \frac{1}{\delta_1}$.

$$(64) \quad 4\left[\delta_1^4\tau_1\tau_2^3 + \delta_1^3\delta_2^2(\tau_1^2+\tau_2^4) + \delta_2^4\tau_1^3\tau_2\right].$$

Let $\delta_1 = \frac{1}{\sqrt{\tau_1\tau_2}}\delta_1'$, $\delta_2 = \frac{\tau_2}{\sqrt{\tau_1\tau_2}}\delta_2'$.

Then

$$(65) \quad \delta = i\sqrt{\tau}\,\delta'.$$

Applying the transformation for δ_1 and δ_2 in (64) The result is

$$(66) \quad 4\left[\delta_1'^4\tau_1^2\tau_2^2 - \delta_1'^2\delta_2'^2(\tau_1^4+\tau_2^4) + \delta_2'^4\tau_1^2\tau_2^2\right].$$

But (66) is the same form in δ' and τ as that obtained in terms of t and τ (under §11). Hence

$$(67) \quad \tau_1^2\tau_2^2(\delta_1'^4+\delta_2'^4) - (\tau_1^4+\tau_2^4)\delta_1'^2\delta_2'^2 = 2(\delta'\tau)^3 f_\tau^3 f_{\delta'}^3$$

where $f_\tau^3 f_{\delta'}^3$ is the third polar of

$$(68) \quad f_{\delta'}^6 \equiv \delta_1'\delta_2' - \delta_1'\delta_2'^5$$

For the roots of the octahedral sextic (68) corresp-

any values of δ can be determined by means of (63).
For $\delta = 0, \infty, 1, -1$. These four values of δ give merely the singular planes on H_1 (and Ω).
For $\delta' = \pm 1$, $\delta = \pm i\sqrt{\tau}$; for $\delta' = \pm i$, $\delta = \mp\sqrt{\tau}$.

If the radical in (63) vanishes, the corresponding points on Ω_4 coincide. The values of τ for the roots of $(\delta\tau)_{\delta\tau}^{3}{}^{3} = 0$.

$$\text{If } \delta' = \pm\tau, \quad \delta = \pm i\tau\sqrt{\tau};$$
$$\delta' = \pm\frac{1}{\tau}, \quad \delta = \pm\frac{i}{\tau}\sqrt{\tau}. \tag{64}$$

Evidently then in general δ is a function of the parameter τ.

§ 13. Quadrics Generated when Ω_4 has Been Projected in an Axis of Perfectivity.

We have already been noticed the collineation $C_1 C_2$ (211), a one harmonic perfectivity — two skew lines H_1 and B_1; i.e. a pair of points on Ω_4 which are transformed into each other by $C_1 C_2$ lie in a line cutting H_1 and B_1 in

points b_2 he runs with the given pair. If the lines forming corresponding points of Ω_4 above on a quadric of the pencil given by (8) and (9), then it can be proved that H_1 and B_1 lie on the same quadric. For H_1 and B_1 each intersect Ω_4 in two points (vide § (2)). These points are on every quadric of the pencil since Ω_4 is the intersection of (8) and (9). But H_1 and B_1 each have a point, p and p respectively, in common with the lines forming corresponding points. Hence if the lines lie on a quadric H_1 and B_1 lie on the same quadric, since each has three points in common with the quadric, and must therefore lie wholly in the surface. Since this proof depends only upon the characteristics of (27) which are common to the twelve eliminants (27)-(38), it is natural to expect to find twelve such pairs of them on any, but in general different, quadric obtainable by elimination

The determination of the quadrics is not difficult. By a reference to the Table of singular points given in § 10 of this chapter, it is clear that (27) transforms $\mathfrak{z}_1$ into $\mathfrak{z}_6$ and vice versa. If as in § 12 a pencil of planes on H_1 is considered, it must be possible to determine δ so that the corresponding plane cuts Q_4 in the points $\mathfrak{z}_1$ and $\mathfrak{z}_6$. If the coordinates of $\mathfrak{z}_1$ are substituted in (61) the resulting value of δ is

$$\delta = \frac{i\sqrt{c}}{\sqrt{a}-\sqrt{b}} \tag{70}$$

The planes of the pencil on $\mathfrak{z}_1$ and $\mathfrak{z}_6$ are

$$y_0 + y_1 + \frac{i\sqrt{c}}{\sqrt{a}-\sqrt{b}}(y_2 + y_3) = 0. \tag{71}$$

The point in which the line joining $\mathfrak{z}_1$ and $\mathfrak{z}_6$ cuts B_1 can be found by getting the intersection of B_1 and the plane (71). This point is $(-i\sqrt{c}, -i\sqrt{c}, \sqrt{a}-\sqrt{b}, \sqrt{a}+\sqrt{b})$. It must lie on any quadric containing the line joining $\mathfrak{z}_1$ and $\mathfrak{z}_6$. By requiring this point to lie on a quadric of the pencil

$$a y_0^2 + b y_1^2 + c y_2^2 + \text{etc.}(a y_0^2 + b y_1^2 + c y_3^2) = 0 \tag{72}$$

it is found that the preceding is to

(73) $\omega = -1$

from the quadric

(74) $y_0^2(a-b)+y_1^2(b-a)+cy_2^2-cy_3^2=0$

contains β_1, β_1 and the lines joining corresponding points of Ω_4, which are transformed into each other by $S_1 S_2$, (27). Since the intersections of C_1 and D_1 with the plane (71) are on the quadric (74), C_1 and D_1 lie on that quadric. This is interchanged with the curves S_1 and S_6, corresponding points under (27) are transformed by (28) into S_2 and S_4 which also are corresponding points under (27). In other words (28) permutes the generators joining corresponding points under (27). It is not difficult to see that these joining corresponding points under (28) lie on the same quadric (74).

By the same method four more quadrics can be obtained and the results can be tabu-

Noted.

Values of u.	Incident at
-1	A_1, B_1, C_1, D_1.
1	A_4, B_4, C_4, D_4.
$\frac{ab}{c-a}$	A_2, B_2, C_2, D_2.
$\frac{-b}{a+c}$	A_5, B_5, C_5, D_5.
$\frac{-b+ic}{a}$	A_3, B_3, C_3, D_3.
$\frac{b+ic}{-a}$	A_6, B_6, C_6, D_6.

(65)

If the values of u are paired off as follows:

$u = \pm 1;\ \frac{b}{c-a},\ \frac{-b}{c+a};\ \frac{-b+ic}{+a},\ \frac{-b+ic}{a}$.

the coordinates become pairs as under an inversion. They therefore form an octahedral group

(76) $$u^6 + 2u^5\left(\frac{a^2+b^2}{ab}\right) + 3u^4 - 3u^2 - 2u\left(\frac{a^2+b^2}{ab}\right) - 1$$

§14. Projection of Ω_4 by a Pencil of Planes on an Edge of T_i.

Each $T_{i\,(i=1,\cdots 6)}$, the fixed tetrahedron of the

sip cycles y_0's has two of its edges in common with T_1 the self-polar tetrahedron of the system is projection of D_4 by a pencil of planes on each edge of T_1 will be considered in the section

Under the subgroup B_1 the two edges to be considered are

$$L_1' \begin{cases} y_1 = 0 \\ y_2 = 0 \end{cases} \text{ and } L_1'' \begin{cases} y_2 = 0 \\ y_3 = 0 \end{cases}.$$

Every plane on L_1' cuts D_4 in four points, none of which lie on L_1' (under B_1). Each plane on cuts L_1'' in only one point since L_1' and L_1'' are skew. Hence there is a four-to-one correspondence between the points of D_4 and the points of L_1''. The four points of D_4 separate into two pairs, each pair lying on a line through the intersection of L_1' and the plane of projection. The relation that separates the pairs and the one that separates the members of a pair can be determined for any particular subgroup B_i ($i \neq 1$)

(77) $\quad y_0 + \lambda y_1 = 0$

is a pencil of planes on L_1'. The intersections of the plane (77) and the curve given by (8) and (11) are the four points

$$\pi_1 : (\lambda \; i\sqrt{c}, \; -i\sqrt{c}, \; \sqrt{a\lambda^2 + b}, \; \sqrt{a + b\lambda^2})$$
$$\pi_2 : (-\lambda \; i\sqrt{c}, \; i\sqrt{c}, \; \sqrt{a\lambda^2 + b}, \; \sqrt{a + b\lambda^2})$$
$$\pi_3 : (\lambda \; i\sqrt{c}, \; -i\sqrt{c}, \; -\sqrt{a\lambda^2 + b}, \; \sqrt{a + b\lambda^2})$$
$$\pi_4 : (-\lambda \; i\sqrt{c}, \; i\sqrt{c}, \; -\sqrt{a\lambda^2 + b}, \; \sqrt{a + b\lambda^2}).$$

The intersection of (77) and L_1'' is the point $\pi_1(-1, 1, 0, c)$. Now the planes

(78) $\quad y_2\sqrt{a + b\lambda^2} - y_3\sqrt{a\lambda^2 + b} = 0$

is on π, π_1, π_2. Hence π_1 and π_2 are projected into π by a line which is the intersection of (77) and (78). Similarly π_3 and π_4 are projected into π by a line which is the intersection of (77) and the plane given by

(79) $\quad y_2\sqrt{a + b\lambda^2} + y_3\sqrt{a\lambda^2 + b} = 0.$

Hence the radical $\sqrt{a\lambda^2 + b}$ separates pairs and $i\sqrt{c}$ separates members of a pair

By using the results that have just been obtained it is possible to deduce the equations of the lines on the singular points, which have been mentioned in §10 of this chapter. If $\delta = 0$, (77) becomes $y_0 = 0$, the plane which intersects with Δ_4 on the singular points S_1, S_2, S_3, S_4. (78) and (79) become

(80) $\quad y_2\sqrt{a} - y_3\sqrt{b} = 0 \qquad$ and

(81) $\quad y_2\sqrt{a} + y_3\sqrt{b} = 0$

respectively. If we look at the coordinates of the singular points (vide §10), shows that S_1 and S_4 lie on the line

(83) $\quad \begin{cases} y_0 = 0 \\ y_2\sqrt{a} + y_3\sqrt{b} = 0, \end{cases}$

and S_2 and S_3 lie on the line

(85) $\quad \begin{cases} y_0 = 0 \\ y_2\sqrt{a} - y_3\sqrt{b} = 0 \end{cases}$

(82) and (85) are lines on $y_1 = 0$.

Similar results in the case of other singular

sheets are touching at and for $t = \infty$

S_5 and S_8 are on the line

$$(84) \quad \begin{cases} \eta_1 = 0 \\ -\eta_2\sqrt{b} + \eta_3\sqrt{a} = 0 \end{cases}$$

and S_6 and S_7 are on

$$(85) \quad \begin{cases} \eta_1 = 0 \\ \eta_2\sqrt{b} - \eta_3\sqrt{b} = 0 \end{cases}.$$

(84) and (85) are on $\eta_0 = 0$.

For $d = \frac{1}{c}\sqrt{\frac{b}{a}}$

S_4 and S_{10} lie on the line

$$(86) \quad \begin{cases} \eta_2 = 0 \\ i\sqrt{a}\,\eta_0 + \sqrt{a}\,\eta_1 = 0. \end{cases}$$

For $d = -\frac{1}{c}\sqrt{\frac{a}{c}}$

S_{11} and S_{12} lie on the line

$$(87) \quad \begin{cases} \eta_2 = 0 \\ i\eta_0\sqrt{a} - \eta_1\sqrt{b} = 0 \end{cases}$$

(86) and (87) are on $\eta_3 = 0$.

For $d = \frac{1}{c}\sqrt{\frac{a}{b}}$

S_{15} and S_{16} lie on the line

(88) $\begin{cases} y_3 = 0 \\ y_0 i\sqrt{b} + y_1\sqrt{a} = 0 \end{cases}$

For $d = -\frac{1}{i}\sqrt{\frac{a}{b}}$

S_{13} and S_{14} lie in the line

(89) $\begin{cases} y_3 = 0 \\ y_0 i\sqrt{b} - y_1\sqrt{a} = 0 \end{cases}$

(88) and (89) are on $y_2 = 0$.

In the case $d = \frac{1}{i}\sqrt{\frac{y_2}{x_2}}$, the corresponding plane of the pencil (77) intersects Q_4 in only two points since for this value of d, $\pi_1 = \pi_3$ and $\pi_2 = \pi_4$; i.e. the two lines of projection have coincided. The same thing is true for $d = -\frac{1}{i}\sqrt{\frac{b}{a}}$. For $d = \pm\frac{1}{i}\sqrt{\frac{a}{b}}$, $\pi_1 = \pi_4$ and $\pi_2 = \pi_3$. Hence planes having double contact with Q_4 have been picked out by these values of d.

If Q_4 is projected by a pencil of planes on a "The singular points pass off in exactly the same ways as when the pencil

on L_1'. Therefore L_1' and L_1'' must as the basis of pencils of projecting planes, project the singular points by the same lines.

L_1' and L_1'' are the principal lines of C_2^2. If C_2^2 is applied to π, π_1 becomes π_2 and vice versa, π_3 becomes π_4 and vice versa. This is consistent with the results of the section.

Analogous reasoning will give the projections of the remaining lines on the singular points. L_2' and L_2'' are the principal lines of C_3^2. Let the pencil of planes on L_2' be

(10) $\quad y_2 + \lambda y_3 = 0$.

Then for $\lambda = 0$, the line

(11) $\begin{cases} y_0 = 0 \\ i\sqrt{a}\, y_1 + \sqrt{c}\, y_3 = 0 \end{cases}$ is on S_2 and S_4;

(12) $\begin{cases} y_0 = 0 \\ i\sqrt{a}\, y_1 - \sqrt{c}\, y_3 = 0 \end{cases}$ is on S_1 and S_3.

For $\lambda = \infty$, the lines

(43) $\begin{cases} y_2 = 0 \\ i\sqrt{c}\,y_1 + \sqrt{a}\,y_3 = 0 \end{cases}$ is on S_{10} and S_{11};

(44) $\begin{cases} y_2 = 0 \\ i\sqrt{c}\,y_1 - \sqrt{a}\,y_3 = 0 \end{cases}$ is on S_9 and S_{12}.

For $\lambda = \pm i\sqrt{\frac{c}{a}}$, the lines

(45) $\begin{cases} y_1 = 0 \\ \sqrt{a}\,y_0 + i\sqrt{c}\,y_2 = 0 \end{cases}$ is on S_5 and S_6;

(46) $\begin{cases} y_1 = 0 \\ \sqrt{a}\,y_0 - i\sqrt{c}\,y_2 = 0 \end{cases}$ is on S_8 and S_7.

For $\lambda = \pm i\sqrt{\frac{a}{c}}$ the lines

(47) $\begin{cases} y_3 = 0 \\ \sqrt{c}\,y_0 + i\sqrt{a}\,y_2 = 0 \end{cases}$ is on S_{13} and S_{14};

(48) $\begin{cases} y_3 = 0 \\ \sqrt{c}\,y_0 - i\sqrt{a}\,y_2 = 0 \end{cases}$ is on S_{15} and S_{16}.

Next consider L_3 and L_5 the first lines of C_3^{11}, C_5^{11}. A pencil of planes on L_3 is given by

(49) $y_0 + \lambda y_3 = 0$.

For $\lambda = 0$, the line

(100) $\begin{cases} y_0 = 0 \\ i\sqrt{b}\, y_1 + \sqrt{c}\, y_2 = 0 \end{cases}$ i.e. on S_1 and S_2;

(101) $\begin{cases} y_0 = 0 \\ i\sqrt{b}\, - \sqrt{c}\, y_2 = 0 \end{cases}$ i.e. on S_3 and S_4.

For $\lambda = \infty$, the line

(102) $\begin{cases} y_3 = 0 \\ \sqrt{c}\, y_1 + \sqrt{b}\, y_2 = 0 \end{cases}$ i.e. on S_4 and S_5;

(103) $\begin{cases} y_3 = 0 \\ \sqrt{c}\, y_1 - \sqrt{b}\, y_2 = 0 \end{cases}$ i.e. on S_{13} and S_{16}.

For $\lambda = \pm\sqrt{\frac{c}{a}}$, the line

(104) $\begin{cases} y_2 = 0 \\ \sqrt{c}\, y_0 + \sqrt{b}\, y_3 = 0 \end{cases}$ i.e. on S_{10} and S_{12};

(105) $\begin{cases} y_2 = 0 \\ \sqrt{c}\, y_0 - \sqrt{b}\, y_3 = 0 \end{cases}$ i.e. on S_9 and S_{11}.

For $\lambda = \pm i\sqrt{\frac{c}{a}}$ the line

(106) $\begin{cases} y_1 = 0 \\ \sqrt{b}\, y_0 + i\sqrt{c}\, y_3 = 0 \end{cases}$ i.e. on S_7 and S_8;

(107) $\begin{cases} y_1 = 0 \\ \sqrt{b}\, y_0 - i\sqrt{c}\, y_3 = 0 \end{cases}$ i.e. on S_6 and S_5.

The list of the twenty-four lines in the negative period is now complete. It remains to deal from the equations of any particular line, or which vertex of T, the triangular tetrahedron, it lies.

Biographical Note

Bessie Young Miller was born in Baltimore, Maryland, in 1884. After graduating from the Girls Latin School, Baltimore, she entered Goucher College from which she received the A.B. degree in 1907. As Fellow of Goucher College she spent the year 1907-1908 as a graduate student in Mathematics and Physics at the University of Chicago. In October 1908 she entered Johns Hopkins University as a graduate student in Mathematics, Astronomy, and Physical Geography. She taught at Bryn Mawr Summer School in 1911-1913. At Johns Hopkins University she was University Fellow in Mathematics for the year 1913-1914.

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